

A simple evolutionary game with feedback between perception and reality

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We study an evolutionary game of chance in which the probabilities for different outcomes (e.g., heads or tails) depend on the amount wagered on those outcomes. The game is perhaps the simplest possible probabilistic game in which perception affects reality. By varying the ‘reality map’, which relates the amount wagered to the probability of the outcome, it is possible to move continuously from a purely objective game in which probabilities have no dependence on wagers, to a purely subjective game in which probabilities equal the amount wagered. The reality map can reflect self-reinforcing strategies or self-defeating strategies. In self-reinforcing games, rational players can achieve increasing returns and manipulate the outcome probabilities to their advantage; consequently, an early lead in the game, whether acquired by chance or by strategy, typically gives a persistent advantage. We investigate the game both in and out of equilibrium and with and without rational players. We introduce a method of measuring the inefficiency of the game, and show that the inefficiency decreases slowly in the approach to equilibrium (for large t it is a power law $t^{-\gamma}$ with $0 \leq \gamma \leq 1$, depending on the subjectivity of the game).

I. INTRODUCTION

A. Motivation

To capture the idea that objective outcomes depend on subjective perception Keynes used the metaphor of a beauty contest in which the goal of the judges is not to decide who is most beautiful, but rather to guess which contestant will receive the most votes from the other judges [12]. Economic problems typically have both purely objective components, e.g., how much revenue a company creates, as well as subjective components, e.g., how much revenue investors collectively *think* it will create. The two are inextricably linked: subjective perceptions alter investment patterns, which affect objective outcomes, which in turn affect subjective perceptions.

To study this problem this paper introduces a simple probabilistic game in which the probability of outcomes depends on the

amount bet on those outcomes. The game introduces the idea of a ‘reality map’ that mediates between subjective perception and objective outcome. This goes beyond Keynes, in that subjective perception actually alters the reality (in his example the faces of the contestants). The form of the reality map can be tuned to move continuously from purely objective to purely subjective games, and different levels of feedback from perception to reality are easily classified and studied.¹

Consider a probabilistic event, such as a coin toss or the outcome of a horse race. Now suppose that the odds of the outcomes of the event depend on the amount wagered on them. In the case of a coin toss, this means that the probability of heads is a function (the ‘reality map’) of the amount bet on heads. For a purely objective event, such as

¹ The results in this paper are presented in more detail in reference [7].

the toss of a fair coin, the reality map is simple: the probability of heads is $1/2$, independent of the amount bet on it. But most events are not fully objective. In the case of a horse race, for instance, a jockey riding a strongly favored horse may make more money if he secretly bets on the second most favored horse and then intentionally loses the race. This is an example of a self-defeating map from perception to reality: if jockeys misbehave, then as the horse becomes more popular, the objective probability that it will win decreases. Alternatively, in an economic setting, if people like growth strategies they will invest in companies whose prices are going up, which in turn drives prices further up. This is an example of a self-reinforcing reality map.

B. Review of related work

There has been considerable past work on situations where subjective factors influence objective outcomes. Some examples include Hommes's studies of cobweb models [9, 10], studies of increasing returns [2], Arthur's El Farol model and its close relative the minority game [3, 6], Blume and Easley's model of the influence of capital markets on natural selection in an economy [4, 5], and Akiyama and Kaneko's example of a game that changes due to the players' behaviors and states [1]. The model we introduce here has the advantage of being very general yet very simple, providing a tunable way to study this phenomenon under varying levels of feedback.

II. GAME DEFINITION

A. Wealth dynamics

Let N agents place wagers on L possible outcomes. In the case of betting on a coin, for example, there are two outcomes, heads and tails. Let s_{il} be the fraction of the i -th player's wealth w_i that is wagered on the l -th outcome. The vector (s_{i1}, \dots, s_{iL}) is the

i -th player's *strategy*, and $p_{il} = s_{il}w_i$ is the amount of money bet on the l -th outcome by player i . Let $p_l = \sum_i p_{il}$ be the total wager on the l -th outcome. If the winning outcome is $l = \lambda$, the payoff $\pi_{i\lambda}$ to player i is proportional to the amount that player bets and inversely proportional to the total amount everyone bets, i.e.

$$\pi_{i\lambda} = \frac{p_{i\lambda}}{p_\lambda} = \frac{s_{i\lambda}w_i}{p_\lambda}.$$

This corresponds to what is commonly called pari-mutuel betting. We assume no "house take", i.e. a fair game. Assume $\sum_l s_{il} = 1$, i.e. that each player bets all her money at every iteration of the game, typically betting non-zero amounts on each possible outcome. The total wealth is conserved and is normalized to sum to one,

$$\sum_i w_i = \sum_{i,l} p_{il} = 1.$$

We will call q_l the probability of outcome l , where $\sum_l q_l = 1$. The expected payoff $\mathbb{E}\pi_i$ is

$$\mathbb{E}\pi_i = \sum_l q_l \pi_{il}.$$

If the vector q is fixed, after playing the game repeatedly for t rounds the wealth updating rule

$$w_i^{(t+1)} = \frac{s_{i\lambda}w_i^{(t)}}{p_\lambda}$$

is equivalent to Bayesian inference, where the initial wealth $w_i^{(t)}$ is interpreted as the prior probability that $q_\lambda = s_{i\lambda}$ and the final wealth $w_i^{(t+1)}$ is its posterior probability. In Bayesian inference, models whose predictions match the actual probabilities of outcomes accrue higher *a posteriori* probability as more and more events occur. Here players whose strategies more closely match actual outcome probabilities accrue wealth on average at the expense of players whose strategies are a worse match.

B. Strategies

We first study fixed strategies. For convenience we restrict the possible number of outcomes to $L = 2$, so that we can think of this as a coin toss with possible outcomes heads and tails. Because the players are required to bet all their money on every round, $s_{i1} + s_{i2} = 1$, we can simplify the notation and let $s_i = s_{i1}$ be the amount bet on heads — the amount bet on tails is determined automatically. Similarly $q = q_1$ and $p = p_1$. The space of possible strategies corresponds to the unit interval $[0, 1]$. We will typically simulate N fixed strategies, $s_i = i/(N - 1)$, where $i = 0, 1, \dots, N - 1$. Later on we will also add rational players, who know the strategies of all other players and dynamically adapt their own strategies to maximize a utility function.

C. Reality maps

The game definition up to this point follows the game studied by Cover and Thomas [8]. We generalize their game by allowing for the possibility that the objective probability q for heads is not fixed, but rather depends on the net amount p wagered on heads. The *reality map* $q(p)$, where $0 \leq q(p) \leq 1$, fully describes the relation between bets and outcomes. We restrict the problem slightly by requiring that $q(1/2) = 1/2$. We do this to give the system the chance for the objective outcome, as manifested by the bias of the coin, to remain constant at $q = 1/2$. We begin by studying the case where $q(p)$ is a monotonic function, which is either nondecreasing or nonincreasing. Letting $q'(p) = dq/dp$, we distinguish the following possibilities:

- *Objective.* $q'(p) = 1/2$, i.e. it is a fair coin independent of the amount wagered. (Other values of $q = \text{constant}$ are qualitatively similar to $q = 1/2$.)
- *Self-defeating.* $q'(p) < 0$, e.g. $q(p) = 1 - p$. In this case the coin tends to oppose the collective perception, e.g.

if people collectively bet on heads, the coin is biased toward tails.

- *Self-reinforcing.* $q'(p) > 0$. The coin tends to reflect the collective perception, e.g. if people collectively bet on heads, the coin becomes more biased toward heads. A special case of this is *purely subjective*, i.e. $q(p) = p$, in which the bias simply reflects people's bets.

It is convenient to have a one parameter family of reality maps that allows us to tune from objective to self-reinforcing. We choose the family

$$q_\alpha(p) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{\pi\alpha(p - \frac{1}{2})}{1 - (2p - 1)^2}. \quad (1)$$

The parameter α is the slope at $p = 1/2$. When $\alpha = 0$, $q(p)$ is constant (purely objective), and when $\alpha > 0$, $q(p)$ is self-reinforcing. The derivative $q'_\alpha(1/2)$ is an increasing function of α ; when $\alpha = 1$, $q'(1/2) = 1$, and $q(p)$ is close to the identity map.² We study the self-defeating case separately using the map $q(p) = 1 - p$.

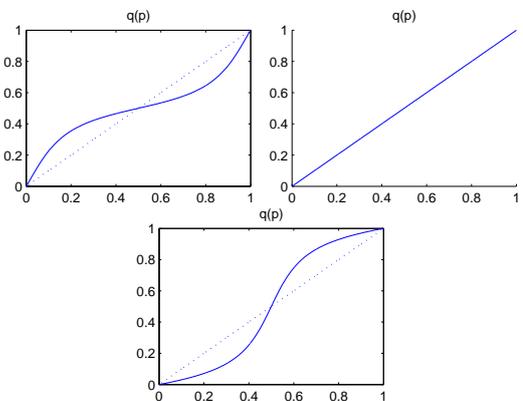


FIG. 1: $q_\alpha(p)$ with different values of parameter: (a) $\alpha = 1/3$; (b) $\alpha = 1$; (c) $\alpha = 3$.

² An inconvenient aspect of this family is that $q_\alpha(p)$ does not contain the function $q(p) = p$. However, $q_1(p)$ is very close to $q(p) = p$ (the difference does not exceed 0.012, with the average value less than half of this). Still, to avoid any side effects, we study the purely subjective case using $q(p) = p$.

III. DYNAMICS OF THE OBJECTIVE BIAS

In this section we study the dynamics of the objective bias of the coin, which is the tangible reflection of “reality” in the game. This also allows us to get an overview of the behavior for different reality maps $q(p)$. We use $N = 29$ agents each playing one of the 29 equally spaced strategies on $(0, 1)$: $1/30, 2/30, \dots, 29/30$, and begin by giving them all equal wealth. We then play the game repeatedly and plot the bias of the coin $q^{(t)}$ as a function of time. This is done several times to get a feeling for the variability vs. consistency of the behavior of different reality maps, as shown in Figure 2.

For the purely objective case $q(p) = 1/2$, the result is trivial. For the self-defeating case, $q(p) = 1 - p$, the results become more interesting, as shown in (a). Initially the bias of the coin varies considerably, with a range that is generally about $0.3 - 0.7$, but it eventually settles into a fixed point at $q = 1/2$. For this case the bias tends to oscillate back and forth as it approaches its equilibrium value. Suppose, for example, that the first coin toss yields heads; after this toss, players who bet more on heads possess a majority of the wealth. At the second toss, because of the self-defeating nature of the map, the coin is biased towards tails. As a result, wealth tends to shift back and forth between heads and tails players before finally accruing to players who play the ‘sensible’ unbiased strategy.

We then move to the weakly self-reinforcing case using equation (1) with $\alpha = 1/2$, as shown in (b). The behavior is similar to the previous case, except that the fluctuations of $q^{(t)}$ are now larger. At the end of 2000 rounds of the game, the bias is much less converged on $q = 1/2$. The bias is also strongly autocorrelated in time — if the bias is high at a given time, it tends to remain high at subsequent times. (This was already true for the self-defeating case, but is more

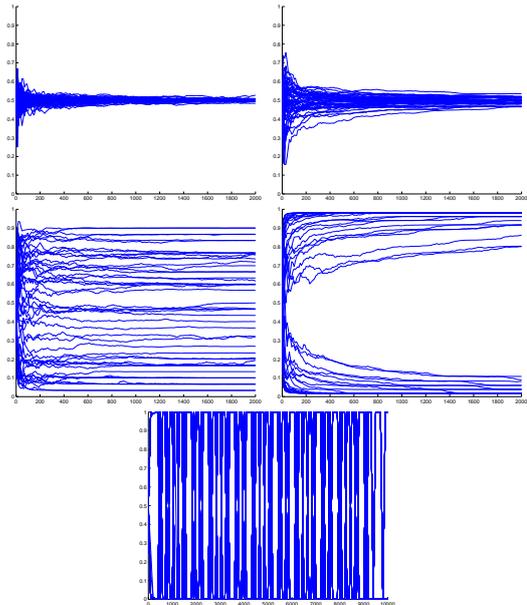


FIG. 2: The objective bias of the coin $q^{(t)}$ as a function of time, for different random number seeds. (a) $q(p) = 1 - p$; (b) in equation (1), $\alpha = 1/2$; (c) $q(p) = p$; (d) $\alpha = 1.5$; (e) $q(p) = 3p \bmod 1$. (a)–(d) 50 runs up to $t = 2000$; (e) 100 runs up to $t = 10000$.

pronounced here). Although this is not obvious from this figure, after a sufficiently long period of time all trajectories eventually converge to $q = 1/2$.

Next we study the purely subjective case, $q(p) = p$, as shown in (c). In this case the bias fluctuates wildly in the early rounds of the game, but it eventually converges to one of the strategies s_i , corresponding to the player who eventually ends up with all the wealth.

As we increase $\alpha > 1$, as shown in (d), the instability becomes even more pronounced. The bias initially fluctuates near $q = 1/2$, but it rapidly diverges to fixed points either at $q = 0$ or $q = 1$. Which of the two fixed points is chosen depends on the random values that emerge in the first few flips of the coin; initially the coin is roughly fair, but as soon as a bias begins to develop, it is rapidly reinforced and it locks in. The extreme case occurs when $q(p)$ is a step function, $q(p) = 0$

for $0 \leq p < 1/2$, $q(1/2) = 1/2$ and $q(p) = 1$ for $1/2 < p \leq 1$. In this case the first coin flip determines the future dynamics entirely: If the first coin flip is heads, then players who favor heads gain wealth relative to those who favor tails, and the coin forever after yields heads, until all the wealth is concentrated with the player that bets most heavily on heads. (And vice versa for tails).

Finally in (e) we show an example of the bias dynamics for the multi-modal map $q(p) = 3p \bmod 1$. In this case the bias oscillates between $q = 0$ or $q = 1$, with a variable period that is the order of a few hundred iterations. We explain this behavior at the end of the next section.

IV. WEALTH DYNAMICS

How do the wealths of individual players evolve as a function of time? The purely objective case $q = \text{constant}$ with fixed strategies and using a bookmaker instead of parimutuel betting was studied by Kelly [11] and summarized by Cover and Thomas [8]. Assuming all the strategies are distinct, they show that the agent with the strategy closest to q asymptotically accumulates all the wealth. Here “closeness” is defined in terms of the Kullback-Leibler distance between the strategy s_{il} and the true probability q_l .

For all reality maps $q(p)$ that we have studied we find that one player asymptotically accumulates nearly all the wealth. As a particular player becomes more wealthy, it becomes less and less likely that another player will ever overtake this player. This concentration of wealth in the hands of a single player is the fundamental fact driving the convergence of the objective bias dynamics to a fixed point, as observed in the previous section. The reason is simple: once one player has all the wealth, this player completely determines the odds, and since her strategy is fixed, she always places the same bets.

It is possible to compute the distribution of wealth after t steps in closed form for the

purely subjective case, $q(p) = p$. The probability that heads occurs m times in t steps is a sum of binomial distributions, weighted by the initial wealths $w_i^{(0)}$ of the players,

$$P_m^{(t)} = \sum_{j=1}^N w_j^{(0)} \left[\binom{t}{m} s_j^m (1 - s_j)^{t-m} \right],$$

and the corresponding wealth of player i is

$$w_i^{(t)} = \frac{s_i^m (1 - s_i)^{t-m} w_i^{(0)}}{\sum_j s_j^m (1 - s_j)^{t-m} w_j^{(0)}}.$$

When the initial wealths are evenly distributed among the players, no player has an advantage over any other. However, as soon as the first coin toss happens, the distribution of wealth becomes uneven. Wealthier players have an advantage because they have a bigger influence on the odds, so the coin tends to acquire a bias corresponding to the strategies of the dominant (i.e. initially lucky) players. Figure 3 shows the probability $P_m^{(t)}$ for $t = 1000$ and $t = 10^5$. After 1000 steps the binomial distributions are still strongly overlapping, and there is still a reasonable chance to overtake the winning strategy. After 10^5 steps, however, the bias of the coin has locked onto an existing strategy s_i , due to the fact that this strategy has almost all the wealth. Once this happens, the probability that this will ever change is extremely low.

We now explain the peculiar bias dynamics observed for the multi-modal map $q(p) = 3p \bmod 1$ in Figure 2(e), in which the bias of the coin oscillates wildly between 0 and 1. Through the passage of time the wealth becomes concentrated on strategies near either $s = 1/3$ or $s = 2/3$, corresponding to the discontinuities of $q(p)$. Suppose, for example, that $p = \sum_i w_i s_i$ is slightly greater than $1/3$, where the $q(p)$ map is close to zero. This causes a transfer of wealth toward strategies with smaller values of s until $p = \sum_i w_i s_i < 1/3$. At this point the bias of the coin flips because $q(p)$ is now close to one and the transfer of wealth reverses to favor strategies with higher values of s . Due to

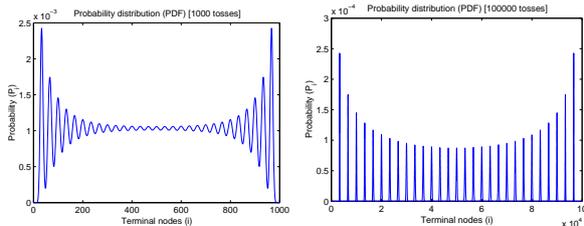


FIG. 3: The probability that heads occurs m times in t rounds of the game with $q(p) = p$, assuming uniform initial wealth, for (a) $t = 1000$ and (b) $t = 10^5$. In (b) note that, although some peaks appear higher than others, the total weight of each peak is the same.

fluctuations in the outcomes of the coin toss this oscillation is not completely regular. It continues indefinitely, even though with the passage of time wealth becomes concentrated more and more tightly around $s = 1/3$. A similar process occurs if the first coin tosses cause convergence around $s = 2/3$. We discuss the initial convergence around $s = 1/3$ or $s = 2/3$ in more detail in the next section.

V. RATIONAL PLAYERS AND NASH EQUILIBRIA

So far we have studied the evolution with a fixed set of strategies. What happens when we instead consider rational players? What are the Nash equilibria? We will see that while the general situation here is quite complicated, we can nonetheless get some analytic insight into the attractors of the game with fixed strategies. Furthermore, we will argue that if one is interested in surviving to dominate the game, the choice of utility function is not arbitrary.

The first question that must be addressed is, “What would rational players reasonably optimize?”. The answer is not obvious. For example, suppose a player maximizes the expected payoff on the next step. Assuming there is higher expectation for the coin to yield heads, the strategy that maximizes the

expected payoff is $s_i = 1$, i.e. betting all the player’s wealth on heads. However, this gives a non-zero probability of bankruptcy, and in the long run guarantees that this player’s wealth will asymptotically be zero. This illustrates the need for risk aversion, and the need to look ahead more than one move. In general looking ahead is very complicated, due to the fact that each player’s move affects not only the player’s expected wealth on the next step, but also the future bias of the coin, and hence the future wealths of all other players.

For the case $q = \text{constant}$ it is possible to show that the strategy that asymptotically accumulates all the wealth maximizes the expected log-return [8]

$$\begin{aligned} \mathbb{E}r_i^{(t+1)} &= \mathbb{E} \log \frac{w_i^{(t+1)}}{w_i^{(t)}} \\ &= \mathbb{E} \log \frac{\pi_i}{w_i^{(t)}} = \sum_l q_l \log \frac{s_{il}}{p_l}. \end{aligned}$$

For $q = \text{constant}$ and fixed strategies, maximizing the log-return repeatedly for one step is equivalent to maximizing it over a long time horizon. For a general reality map $q(p)$, however, this is no longer sufficient; it is easy to produce examples for which two step optimization produces different results than single step optimization, due to the effect of changes in the strategy’s wealth on the future bias of the coin. For the case where the strategy’s wealth is negligible, however, the situation is greatly simplified: It is possible to show that maximizing the one step log-return at each step is equivalent to maximizing the log-return over many steps [7]. In a game with many players, each of whom has small wealth, maximizing the log-return for the next step may be a good approximation to the optimal strategy. As we have already seen, acquiring a lead in the early stages of the game gives an advantage that tends to persist later on.

The expected log return for one step can be written

$$r(s) = q \log \frac{s}{p} + (1 - q) \log \frac{(1 - s)}{(1 - p)},$$

where as before $q = q_1$, $s = s_1$, $w = w_1$, etc. The first derivative is

$$\frac{dr}{ds} = q'w \left[\log s - \log p - \log(1-s) + \log(1-p) \right] + q \left(\frac{1}{s} - \frac{w}{p} \right) + (1-q) \left(-\frac{1}{1-s} + \frac{w}{1-p} \right),$$

where $q' = dq/dp$. A sufficient condition for $dr/ds = 0$ is $s = q = p$. In this case the second derivative is

$$\frac{d^2r}{ds^2} = \frac{1-w}{s(1-s)} \left[2wq' - (1+w) \right].$$

For the map q_α the strategy $s = 1/2$ is a local maximum when $\alpha < (1+w)/(2w)$. Providing this condition holds, this implies that this strategy is what we will call a *myopic log Nash equilibrium*, i.e., it is a strategy that, when played against itself, gives the best possible expected log return for the next round of the game. The myopic log Nash equilibria depend on the reality map $q(p)$, but in general they also depend on the wealth of the players, so that they can be dynamic, shifting with each coin toss. For example, when $\alpha < 1$, $s = 1/2$ is always a myopic log Nash equilibrium, but when $\alpha > 1$ the myopic log Nash equilibria strongly depend on the wealth of the players. This makes long-range optimization difficult.

This is illustrated in Figure 4, where we show the expected log return for a player (arbitrarily labeled “first”) playing a myopic log Nash strategy against another (second) player using a fixed strategy with $s = 1/2$. When the first player’s wealth is low, $1/2$ is the optimal strategy, and it is a myopic Nash equilibrium. As the first player gains in wealth two strategies on either side of $s = 1/2$ become superior; as wealth increases, these strategies become more and more separated from $s = 1/2$, and in the limit as $w \rightarrow 1$ the optimal strategy is either $s = 0$ or $s = 1$.

In the limit as the first player’s wealth gets large the possible myopic log Nash equilibria do a good job of predicting the attractors

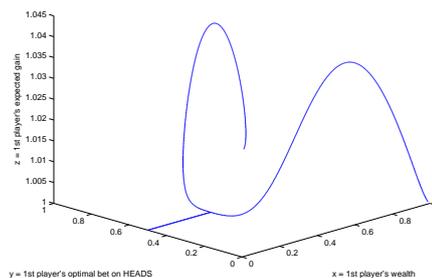


FIG. 4: The expected log-return $r(s)$ for a player (arbitrarily labeled “first”) using a myopic log Nash strategy against a player using a fixed strategy $s = 1/2$, where $q(p) = q_\alpha$ with $\alpha = 2$. This is plotted against the first player’s wealth on the x axis and the first player’s optimal bet on heads on the y axis.

of the objective dynamics shown in Figure 2. For the self-defeating case, $q(p) = 1 - p$, there is a unique myopic log Nash equilibrium at $s = 1/2$, corresponding to the unique attractor for q . For the self-reinforcing case with $\alpha < 1$ this is also true. For the identity map $q(p) = p$ the entire interval $0 \leq s \leq 1$ is a myopic log Nash equilibrium. When $\alpha \gg 1$ either $s = 0$ or 1 can be myopic log Nash equilibria. In each case these correspond to the attractors of the wealth dynamics.

The multimodal map $q(p) = 3p \bmod 1$ is interesting because all the intersections with the identity are local minima for the expected log return. Instead, the system is attracted to the discontinuities of the map at $s = 1/3$ and $s = 2/3$. It is as if one can think of the discontinuities of the map as being connected (with infinite slope), creating intersections with the identity that yield local maxima of the log return. While we do not have a formal method of proving this, we gave an intuitive explanation for how this stability comes about in Section IV.

VI. EFFICIENCY

As the game is played the reallocation of wealth causes the population of players to become more efficient in the sense that there are poorer profit opportunities available for optimal players. This is analogous to financial markets, where wealth reallocation due to profitable vs. unprofitable trading has long been hypothesized to result in a market that is efficient in the sense that all excess profit-making potential has been removed. In financial economics efficiency is taken as a postulate; there is no theory that guarantees that markets out of equilibrium will always be attracted to a perfectly efficient market equilibrium, and no way to quantitatively measure the inefficiency of a market when it is out of equilibrium. Our game provides a simple setting to study the approach to efficiency in an out-of-equilibrium context.

We can measure the inefficiency of our game based on the returns of what we will call a *rational ϵ player*. This player knows the strategies of all other players, and pursues an optimal strategy that maximizes her expected log returns. This player has infinitesimal wealth ϵ , so that her actions have a negligible effect on the outcome of the game. In the purely objective setting where $q = \text{constant}$ the approach to efficiency is guaranteed by the fact that the wealth dynamics are formally equivalent to Bayesian updating, implying all the wealth converges on the correct hypothesis about the bias of the coin. For more general settings this is no longer obvious, as there is no longer such a thing as an objectively correct hypothesis.

We have studied the approach to efficiency numerically for a variety of different reality maps as shown in Figure 5. To damp out the effect of statistical fluctuations from run to run we take an ensemble average by varying the random number seed. For $q(p) = p$ we find that the inefficiency is essentially zero at all times. In every other case we find that the efficiency is a decreasing function of time,

asymptotically converging as a power law $t^{-\gamma}$ with $0 \leq \gamma \leq 1$. For the self defeating case $q(p) = 1 - p$ we observe $\gamma \approx 1$; for other values of α we observe $\gamma < 1$. For example, for $\alpha = 0.5$, $\gamma \approx 0.6$, for $\alpha = 1.5$, $\gamma \approx 0.25$, and for $\alpha = 2$, $\gamma \approx 0.5$. For maps close to the purely subjective case, the inefficiency is initially quite small but convergence is correspondingly slow. For example, compare 5 (a) and (b). The self-defeating case is initially much more inefficient than the mildly self-reinforcing case, but by $t = 10,000$ the situation is reversed. The rate of convergence in efficiency reflects the slow convergence in the bias of the coin to its fixed point attractor.³

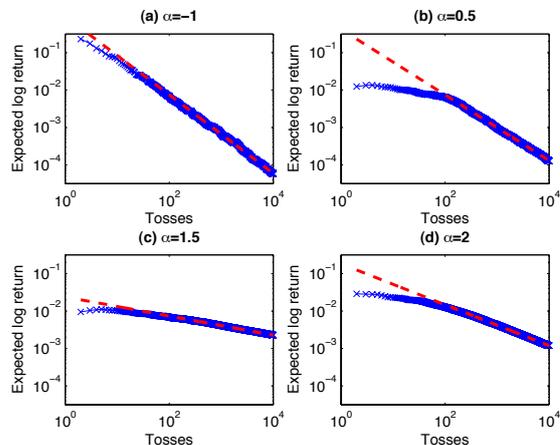


FIG. 5: The inefficiency of the game is measured by the log returns of an optimal ϵ player, shown as a function of time using $N = 3000$ players. Plots are in double logarithmic scale. (a) $q(p) = 1 - p$; (b) $\alpha = 0.5$; (c) $\alpha = 1.5$; (d) $\alpha = 2.0$.

³ We have performed simulations with different numbers of agents and find that domain of validity of the power law scaling is truncated for small N (e.g. for $N \approx 30$ it extends only to roughly $t \approx 1000$). The length of validity of the power law scaling in time increases with N . Thus there is a finite size effect, indicating that the power law scaling is exactly valid only in the limit as $N \rightarrow \infty$ and $t \rightarrow \infty$.

VII. INCREASING RETURNS

When the reality map for a game of chance is purely objective a player can't manipulate the outcome unless she cheats. In contrast, for more general reality maps, under some circumstances the player can use the subjective dependence of $q(p)$ to manipulate the objective odds to her advantage. This manifests itself as increasing returns to scale—as the player acquires more wealth, the ability to manipulate the odds increases, and the expected return also increases.

To measure this we study a rational strategy that exploits its complete knowledge of the strategies of the other players to maximize the expected log-return for the next step. We start with given wealth assignments for the players and vary the fraction of the wealth for the rational player vs. the other players. Under some circumstances, which depend both on $q(p)$ and the distribution of wealth, for self-reinforcing reality maps we find that the returns are an increasing function of the wealth of the rational player. Two examples are given in Figure 6.

VIII. SUMMARY

We have introduced a very simple evolutionary game of chance with the nice property that one can explicitly study the influence of the player's actions on the outcome of the game. By altering the reality map $q(p)$ it is possible to continuously vary the setting from completely objective, i.e. the odds are independent of the players' actions, to completely subjective, i.e. the odds are completely determined by the players' actions.

This is an evolutionary game in the strong sense: Only one player survives to have non-negligible wealth. Our results suggest that the myopic maximization of expected log returns is a fairly good survival strategy, certainly much better than simply maximizing returns. However, in contrast to the purely objective case, it is provably not an optimal

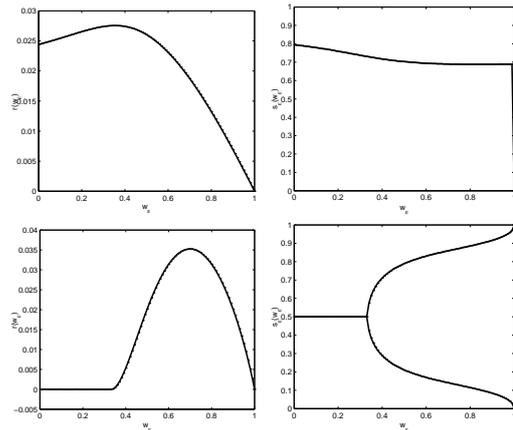


FIG. 6: The left column shows the expected log returns $r(w)$ for an optimal player and the right column shows the corresponding optimal strategies $s(w)$, both as a function of her wealth w . In the top row $\alpha = 1/2$ and $t = 7$, where $w_i^{(7)}$ is produced by starting at $w_i^{(0)} = 1/N$ and generating 7 heads, where $N = 29$. In the bottom row $\alpha = 2$, $t = 0$, and $w_i^{(0)} = 1/N$.

strategy. This is due to the complications induced by the feedback between the success of the players and the objective reality as reflected in the bias of the coin.

It has long been known that subjective effects can play an important role in games, causing problems such as increasing returns and lock-in to a particular outcome due to chance events. This model shows that the existence of subjective effects alone are not enough. Instead, for most of the properties we study here, such as selecting between alternative equilibria or increasing returns, we need the self-reinforcing effects to be sufficiently strong to destabilize the objective dynamics. There is a competition between the stabilizing force of wealth concentration and the destabilizing properties of $q(p)$ when $q' > 0$. This game shows how these effects become steadily stronger as the self-reinforcing nature of the reality map increases. It also shows that these effects are generally complicated and wealth dependent, even in this simple situation. The myopic log Nash equi-

libria are strongly wealth dependent. Since wealth evolves through time, the myopic log Nash equilibria also evolve in a time dependent manner.

This game provides a setting in which to study the progression toward efficiency in an out-of-equilibrium context. We have introduced a notion of efficiency that closely resembles arbitrage efficiency in financial markets. We always observe a progression toward efficiency, except for the purely subjective case, in which it appears that any configuration of player strategies automatically produces an efficient market. This isn't surprising, since in the purely subjective case there is no preferred strategy. In every other case, as wealth is reallocated, the game becomes more efficient, in the sense that there are fewer profit-making opportunities for skillful players. For the examples we observe that the inefficiency as a function of time asymptotically decreases as a power law.

One might consider several extensions of the problem studied here. For example, one could study learning (see e.g. [13]). Another interesting possibility is to allow more general reality maps, in which q is a multidimensional function with a multidimensional

argument that may depend on the bets of individual players. For example, an interesting case is to allow some players, who might be called pundits, to have more influence on the outcome than others. It would also be very interesting to modify the game so that it is an open system, e.g. relaxing the wealth conservation condition and allowing external inputs. This may prevent the asymptotic convergence of all the wealth to a single player, creating more interesting long-term dynamics.

IX. ACKNOWLEDGEMENTS

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