ARBITRAGE, INFORMATION, AND THE COMPETITIVE ORGANIZATION OF DISTRIBUTIONS OF PROFITABILITY

Paulo L. dos Santos and Jangho Yang

27th September 2018


Complexity Economics
Arbitrage, Information, and the Competitive Organization of Distributions of Profitability

Paulo L. dos Santos* and Jangho Yang†‡

September 27, 2018

Abstract

This paper presents evidence that large-sample, cross-sectional distributions of the rate of return on capital (RoC) realized by enterprises across twenty European economies exhibit the same functional form: Double-stretched-exponential distributions. To account for this observation, the paper develops a systemic, economic model of the macroscopic outcomes of the competitive actions of arbitrageurs—who seek profits defined by heterogeneity across values of RoC. The model suggests the observed distributions embody competitively established aggregate tradeoffs between the pecuniary returns and costs arbitrageurs incur. Those costs can be taken as rising on the measure of organization the actions of arbitrageurs impose on distributions of RoC, ensuring competition effectively prices informational gains in ways we can observe. The paper’s discussion defines a series of new, observable, macroscopic measures of the performance of competitive systems. It also points to the aptness of understanding prices as parts of structures of generalized Marx-Sraffa “prices of production,” predicated on the characteristics of capital-market statistical equilibria; to a general theoretical approach to the regulation of certain economic quantities by arbitrage; and to the role the costs of informational gains play in shaping observable outcomes in the operation of certain types of goal-seeking, self-organizing systems.

Keywords: Econophysics, Information Theory, Distributions of Profitability, \((c, d)\) entropies, Observational Economics

1 Introduction

The quest for profitability is the defining organizing principle of decentralized capitalist economies. Enterprises organize all aspects of the production and sale of goods and services so as to maximize

---

*Department of Economics, New School for Social Research, and Corresponding Author.
†Institute for New Economic Thinking at the Oxford Martin School, University of Oxford.
‡Oxford Martin Programme on Technological and Economic Change, Oxford Martin School, University of Oxford.
the rate of return on assets represented by their profits. Realized measures of profitability reflect the functioning of markets for labor, capital, and for the inputs employed and outputs produced by enterprises. Together with expectations about future profitability, those measures also inform the allocation of capital, shaping the extent to which different undertakings are supported by investors. As a result, distributions of profitability should tell us much about the nature and dynamic evolution of competitive capitalist economies.

A number of important contributions have explored different aspects of this expectation. Some drew on theoretical insights to speculate about the shape of those distributions.\[1, 2\] Others sought to investigate the empirical purchase of these and other Classical contentions about the effects of competition on observable moments of distributions of profitability.\[3, 4, 5\] A more recent set of contributions opened a different, observational line of inquiry, centering on efforts to identify the empirical form taken by distributions of profitability as a first step informing the development of theories of competition in capitalist economies.\[6, 7\] Drawing on data from U.S. non-financial corporations, those studies found evidence that the annual, cross-sectional distributions of firm-level profitability are consistently well-approximated by the power-exponential forms given by asymmetric Subbotin distributions. This work led to two initial theoretical contributions—broadly grounded on the Classical contention that capital mobility and arbitrage tend to equate realized measures of profitability at an emergent general rate of profit—that account for those distributions as statistical equilibria conditioned either by specific micro-kinetic, drift-diffusion processes,\[8\] or by the “bounded rationality” of “inattentive” or informationally constrained individual arbitrageurs in capital markets.\[9\]

Grounded on Information Theory and a critical application of theoretical insights from Classical and Austrian Political Economy, this paper offers three contributions to these discussions.

First, it offers a full account of recent, summarily reported results of a study that confirmed, broadened, and extended the finding of persistent formal regularities in distributions of measures profitability.\[10\] Drawing on qualitatively larger ($\sim 10^4 - 10^5$), nationally representative samples of private and publicly traded companies, that study established that country-level cross-sectional distributions of realized firm returns on capital (RoC) are persistently well-described by the generalized power-exponential forms given by double stretched exponential distributions.

The ubiquity of these distributional forms over fundamentally interdependent individual measures of profitability is remarkable. Their strong modality supports the Classical contention that while complex competitive interactions, entrepreneurial innovations, and broader economic change continuously shift realized individual measures of profitability, the mobility of capital and broader technical and marketing efforts toward higher yielding undertakings gives rise to a general rate of return $r_g$ toward which individual measures of profitability are regulated.\[11, 12, 13\] The formal persistence in the distributional tails away from this rate suggest those complex and unobservable processes consistently find the same macroscopic expression. This enables the pursuit of a dis-
tinctively useful analytical approach to the competitive regulation of realized measures of RoC. Instead of offering micro-level theorizations based on strong specifications of unobservable characteristics of individual agents, markets, and interactions, it is possible to consider the persistent distributional patterns we observe as the emergent, systemic outcome of micro-level competitive dynamics that observationally grounded economic theory can and ought to explain.

Second, by developing an explicit informational accounting of complex, dynamic patterns of economic competition and change, the paper shows how the double stretched exponential distributions we observe are expressions of remarkably simple economic and combinatorial facts. The only economic actions in a complex, decentralized profit-driven economy capable of generating, in a statistically meaningful manner, entropy reductions or informational gains in the distribution of profitability are interventions by arbitrageurs: Agents seeking to profit from heterogeneity in that distribution by moving capital value and broader economic efforts toward undertakings and practices associated with higher measures of profitability. All other competitive or entrepreneurial actions, and broader economic developments or changes, are overwhelmingly likely to result in irreversible or disorganizing changes in the distribution.

The observed patterns of organization are thus an emergent, unintended outcome of the unobservable profit-driven actions of arbitrageurs within broader, complex and disorganizing competitive process. Those patterns are shaped by the manner in which the actions of arbitrageurs endogenously establish a general rate of profit given the measures of profitability they observe, and by the economic tradeoff between aggregate arbitrage costs and returns enforced by capital-market competition. Significantly, this tradeoff is shaped by the fact that over any given time period, the aggregate costs of the actions of arbitrageurs are overwhelmingly likely to be increasing on the gross measure of entropy reduction the actions of arbitrageurs effect on distributions of RoC.

The resulting systemic or macroscopic model of competitive arbitrage improves on extant accounts in a number of important ways. It offers an explicit account of the endogenous, competitive emergence of a general rate of profit. This yields insights into the systematic presence of positive entrepreneurial quasi-rents and monopolistic rents in the dynamic, competitive regulation of profitability. Perhaps most significantly, the model leads to an understanding of the stretched exponential regulation of measures of profitability not as the expression of the micro-kinetic evolution or cognitive limits of would-be representative or average individuals, but as an emergent, systemic pricing of information by the competitive search for arbitrage profits.

Third and finally, the paper discusses how the model it advances allows the characterization of the observed distributions of RoC as statistical equilibria, defined over a centrally important domain for decentralized, market economies. This in turn opens the possibility of a characterization of competitive prices as parts of generalized systems of Classical “prices of production” conditioned by wage structures, productive techniques, and statistical equilibria in capital markets. It also offers a general approach to the characterization of observable states of a class of goal-driven,
self-organizing systems are often relevant to economic inquiry.

The paper proceeds as follows. Section 2 presents the evidence we have considered. Section 3 offers a discussion on the methodological implications of macroscopic stability in distributions of profitability, the explanatory burdens it places on theories of competition, and on recent attempts to offer a theoretical account for the central form of regulation of profitability around its modal value. Section 4 offers the conceptual discussion of the economic contentions at the heart of this paper. Section 5 offers a formal statement and solution of the model implied by that discussion. Section 6 concludes with a discussion of lines of further work suggested by the paper.

2 The Distributions and Stretched-Exponential Fits

We used data from the Amadeus Company Information Database,[20] which currently contains data on more than 20 million individual European enterprises. The database contains samples that are representative of the structure of each economy, including firms of all sizes. Approximately 99 percent of them are private, non-corporate organizations.

We constructed national, annual frequency histograms for each year between 2006 and 2015 for the ratio $r_i$ of companies’ “Earnings Before Interest, and Tax” (EBIT) to their “Total Fixed Assets.” We considered 20 economies with more than 2,000 average observations per year during the period in question. This resulted in a set of twenty European economies, including Germany, France, the UK, Italy, Russia, and Spain. We included all non-financial companies with an RoC between -200 and 200 percent, a wide, economically relevant range beyond which observed frequencies are at least two orders of magnitude smaller than those over central values of $r_i$. This resulted in twenty national series, each with ten end-of-year cross sections. Those cross sections are densely populated, with an average of 152,492 enterprises in each.\(^1\)

The cross sections reveal a strikingly consistent pattern of organization, shown in Figure 1, over a coarse graining into bin values $r_k$, for ten of the observed national economies.

\(^1\)The countries with the greatest average number of enterprises were Italy, Spain, and France, with on average 487,313, 456,338, and 446,898 enterprises per year, respectively. Greece and Serbia had the smallest average annual samples, at 13,516 and 31,493, respectively.
Figure 1: Observed Log-Frequencies and Minimum-Kullback-Leibler-Divergence Fits for Stretched-Exponential Model, France, Italy, Sweden, Portugal, Belgium, Germany, Spain, Czech Republic, Romania, and Ukraine. 2006-2015.
For each cross-section, we established remarkably good piecewise fits for stretched exponential models for the distributions of $x_k = r_k - r_g$.

$$f_k = \exp \left[ 1 - (\lambda_n x_k + \mu_n)^{\frac{1}{\alpha_n}} \right], \quad x_k < 0 \ ; \ f_k = \exp \left[ 1 - (\lambda_s x_k + \mu_s)^{\frac{1}{\alpha_s}} \right], \quad x_k > 0$$

(1)

for \(d, \lambda_s, \mu_s, \mu_n > 0\) and \(\lambda_n < 0\). Each of these tails is subject to a normalization constraint given, in a piecewise description of the entire distribution across all \(x_k\) with \(a\) as the proportional mass of the right tail, by,

$$-\lambda_n = \frac{e d_n}{1 - a} \Gamma \left( d_n, \mu_n \right) ; \quad \lambda_s = \frac{e d_s}{a} \Gamma \left( d_s, \mu_s \right)$$

(2)

These implicitly define \(\mu_n = \mu_n(d_n, \lambda_n)\) and \(\mu_s = \mu_s(d_s, \lambda_s)\) over the parameter ranges under consideration.

Using the minimization of the Kullback-Leibler divergence as a criterion for parameter selection, models \(f(d, \lambda)\) based on 1 and 2 offered very good fits for the observed histograms \(h\), as measured by the informational indistinguishability index \(\text{ID}_h[f(d, \lambda)]\) between models and observations.[21, 22]. As reported in Table 2, the estimated models on average account for 0.990 of the informational content in each of the right tails, and 0.985 in the left ones. The average \(\text{minimum}\) informational fit achieved across all countries is 0.982 for right tails, and 0.968 for left tails. Notably, among the three economies with more than 400,000 annual observations on average (France, Italy, and Spain), the fits are even better: The average right-tail \(\text{ID}_h[f(d, \lambda)]\) is 0.996, while the average minimum is 0.992. The average left-tail index is 0.991, with an average minimum value of 0.983.

<table>
<thead>
<tr>
<th>Informational Indistinguishability Indices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belgium</td>
</tr>
<tr>
<td>Right</td>
</tr>
<tr>
<td>min</td>
</tr>
<tr>
<td>0.991</td>
</tr>
<tr>
<td>Germany</td>
</tr>
<tr>
<td>Right</td>
</tr>
<tr>
<td>min</td>
</tr>
<tr>
<td>0.996</td>
</tr>
<tr>
<td>Poland</td>
</tr>
<tr>
<td>Right</td>
</tr>
<tr>
<td>min</td>
</tr>
<tr>
<td>0.995</td>
</tr>
<tr>
<td>Slovakia</td>
</tr>
<tr>
<td>Right</td>
</tr>
<tr>
<td>min</td>
</tr>
<tr>
<td>0.984</td>
</tr>
</tbody>
</table>

Figure 2: Summary Measures of Informational Indistinguishability Between Estimated Stretched-Exponential Models and Observed Histograms.
The evolution of estimated parameter values are depicted in Figures 3 and 4 for the five largest, advanced economies we observed.

Figure 3: Estimated values for $d_n$ and $d_s$, Germany, France, UK, Italy, and Spain, 2006-2015.

Figure 4: Estimated values for $\lambda_n$, $\lambda_s$, and $M$ for each tail, Germany, France, UK, Italy, and Spain, 2006-2015.

The full significance of these findings for economic analysis follows from the fact that the stretched-exponential tails in 1 and 2 maximize generalized $(1, d)$ entropy functionals among all distributions $f$ subject to a first-moment contraint $\langle x_i \rangle_f = M$. The persistent and ubiquitous stretched-exponential fits may thus be understood as more than simply descriptive. They suggest stretched-exponential statistical-equilibrium models of the results of the competitive regulation of returns on capital: That the complex and largely unobservable micro-level patterns of competitive interactions in markets for goods, labor, and capital consistently resolve themselves into two macroscopic regularities, corresponding to the conditions defining stretched-exponential distributions as entropy maxima.

First, competitive interactions give rise to interdependences between individual values of $x_k$ that impose the kinds of reductions in phase-space volume that ensure that the correct, extensive entropy functional describing the statistical weight of a macroscopic state $f$ is given by,

$$S_f = S_d(f) = \sum_{k=1}^{s} e \Gamma (d + 1, 1 - \log f_k) - 1$$ (3)
Second, competition also ensures that the economic system functions as if the average deviation from the modal value $r_d$ in each tail were approximately given.

Based on these observations we argued that the explanatory burden on an economic theory of the competitive regulation of individual values of $x_k$ is to provide an economically meaningful account of the processes capable of generating these two macroscopic regularities. In what follows, we develop such an economic theory.

3 Macroscopic Stability and Theorizing Competition

Before proceeding, a brief methodological discussion motivating and defining the application of information-theoretic methods for analysis of complex economic systems is necessary.

The persistence of a highly organized pattern in distributions of measures of profitability, in large samples, across countries, and over time is striking. Competition ensures individual values of profitability are interdependent. Locally, the effects of efforts by enterprises to boost profitability through technical innovation, through the development of new products, consumers, and markets, and through control over wages and other input costs are dynamically influenced by the analogous actions of other enterprises competitively engaged in the same input and output markets.[18, 25]

More broadly, developed credit and capital markets create long-range interdependences between individual measures of profitability: They enable the general reallocation of capital value, productive capacities, and competitive efforts from low- to high-profitability undertakings, jointly conditioning demand and supply conditions, prices, and profitability across the markets involved.[18, 11, 13]

Finally, at least over some time horizons constraints on aggregate demand for goods and services may create further long-range interdependences between measures of profitability.[26, 27, 28, 29]

The distributions we observe are thus generated by systems composed of very large numbers of non-linearly coupled members. Yet the complex micro-level competitive interactions involving enterprises, investors, workers, and consumers in each of these economies persistently resolve themselves into the simple distributional pattern reported above. The presence of significant measures of macroscopic stability across distributions of a quantity at the very heart of the functioning of decentralized capitalist economies has important implications for economic analysis.

3.1 The Method of Observational Political Economy

Contemporary mainline Economics seeks to characterize competitive processes on the basis of detailed descriptions of individual behavior.[30] Those descriptions are based on strong specifications of the consumption preferences, technological constraints, and knowledge states shaping individual actions, and of the forms taken by competitive interactions.[31] While potentially useful as bases for pursuing thought exercises, there are at least three reasons ensuring this approach offers a poor foundation for observational inquiry into the functioning of economic systems.[14]
First and as well established across a variety of disciplines, detailed descriptions of individual behavior are at best impractical bases to characterize the functioning of large systems composed of many interacting parts. This is true even when the laws or regularities governing individual behavior are very well understood. Economic systems pose an additional and characteristic difficulty relative to physical systems in this regard: all economically relevant features of individuals are themselves shaped by economic competition and broader social interactions. If the characteristics of economic individuals and economic interactions are mutually defining, taking the former as an analytical starting point is not just impractical. It is arbitrary: There is no a priori reason to expect regularities allowing us to develop successful characterizations of the functioning of economic systems to be defined at the level of individual characteristics and behavior.

Second, many of the individual characteristics and behaviors upon which contemporary theorizations of competition are predicated are unobservable, straining the scientific soundness of suppositions made about their nature. And third, the annual or quarterly frequencies at which we are typically able to observe some elements of individual economic states are far lower than the frequencies at which individuals interact. The quantities we can observe reflect not the behavior of individuals per se, but the accumulated result of many interactions among large numbers of individuals. Between the times at which we can take measurements, much of the information about individual behavior has been lost—both as a result of large numbers of interactions, and of changes to the individuals themselves.

Macroscopic stability in distributions of profitability enables the pursuit of a more fruitful analytical path. It strongly suggests the dynamic trajectories of individual measures of profitability are subordinated to a general, emergent pattern of systemic regulation common to all observed economies. Competition appears to impose upon decentralized capitalist economies a consistent form of the emergent self-organization or “spontaneous order” postulated by some of the salient figures in the history of economic thought. In the frequencies of profitability shown above we are in fact observing the manner and extent to which Adam Smith’s “invisible hand” regulates the outcomes of individual competitive efforts. Those frequencies define the explanatory burden of observationally grounded, systemic theories of economic competition.

3.2 Existing Accounts of the Competitive Regulation of Profitability

The formal persistence in distributions of profitability has been the subject of two recent theorizations. Both characterize those distributions as statistical equilibria, but define them at least in part in relation to detailed descriptions of “representative” or average individual dynamic evolutions or individual cognitive constraints. Recourse to the characteristics of an individual in accounting for competitive processes creates a few difficulties.

The first contribution offered a drift-diffusion model of a “representative,” time-homogeneous evolution for individual profitability capable of generating stationary Subbotin distributions.
most significant difficulty with this line of explanation follows from the fact that economic competition is fundamentally about interactions, interdependences, and various dimensions of heterogeneity across competitors. A representation that reduces the outcomes of such processes to large numbers of identical, independent, individual evolutions may be formally successful, but it is not readily evident how those evolutions may be related to economic behavior or functioning.

The second contribution offers a more economically meaningful interpretation, based on the idea of “Quantal-Response Statistical Equilibria” (QRSE) recently developed and applied to distributions of profitability and measures of productivity growth.\[9, 36]\ That account draws on Game Theory and relies on the idea that economic agents have “bounded rationality” or are otherwise “inattentive.”\[37, 38]\ Under this view, distributions of profitability are seen as reflecting the limited capacity for information processing of the typical or average individual, who can only respond probabilistically when considering whether to undertake a profitable arbitrage reallocation of capital. The profitability of an enterprise, in turn, is taken as statistically conditioned by whether it is experiencing entry or exit of competitors in the markets where it operates. The interplay between individual quantal responses to enter or exit a market, and the response of profitability to those decisions, are taken to define negative-feedback processes that tend to keep measures of profitability near $r_g$.

While such an approach can support useful thought exercises, the appeal to well-understood individual cognitive limits creates three related difficulties. First and as noted above, it is not generally possible to recover details of individual stimulus-response behavior from the annual data we observe. It is possible to use the data we observe to infer parameter values in QRSE models of individual behavior and of the response of profitability to it, but those inferences will be inherently shaped by model specifications. Without this important qualification, the results of such inferences may be very difficult to interpret.

Second, competitive entry and exit into markets depend both on the intentions of arbitrageurs and on the degree to which capital markets accommodate and finance them. Only at capital-market equilibria are all intentions accommodated. The degree to which competitive entry increases competition in an industry hinges not on how many individuals choose to enter, but on the measure of productive and competitive resources allocated to that industry. That hinges most generally on the volume of financing would-be arbitrageurs can mobilize. A single, well-funded arbitrageur can have a more dramatic impact on competition and profitability than many poorly funded ones. This is closely related to a final difficulty. In competitive capital markets, individual cognitive limits are likely to be wholly irrelevant to systemic market outcomes. One person’s error or broader inability to undertake an existing arbitrage operation is another person’s opportunity to do so until the potential for arbitrage profits is exhausted.

The observed distributional patterns for measures of profitability are not reducible to the intentions, knowledge, or cognitive limitations of any individual. Observable economic outcomes seldom
are.\textsuperscript{[39]} They are the emergent result of complex competitive interactions across all markets for goods, labor, and capital. We can account for them without any strong specifications of individual evolutions or behavioral characteristics, on the basis of simple economic and combinatorial considerations.

4 Competition, Arbitrage, and Organization

Our account is founded on three analytical elements. First, it approaches the functioning of a competitive economic system with the aid of an informational accounting:\textsuperscript{[40, 41]} A mapping from the wide variety of unobservable complex competitive actions, interactions, and broader changes in economic conditions, to the evolution of the entropy of observable distributions of profitability. This accounting identifies the actions of competitive arbitrageurs as the agency responsible for patterns of organization in those distributions. Second, it considers the endogenous formation by the actions of arbitrageurs of the general rate of return or opportunity cost of capital, \( r_g \), and the macroscopic regularities it imposes on the distributions of \( x_k \). And third, it takes the observed pattern of organization in those distributions as the result of the economic calculus competition and capital-market signals impose on the systemic outcomes of arbitrage operations.

This section discusses these propositions and their formal expressions in turn.

4.1 Informational Accounting of Competitive Processes

Profitability is a social, not a physical measure. The functioning of the economic systems conditioning its values across individual enterprises is primarily driven by profit-seeking behavior, not thermodynamic considerations. Yet the informational and locationally independent measure of heterogeneity given by the entropy of the distribution of profitability captures a functionally relevant moment of the complex patterns of competitive interaction in decentralized, capitalist economies. While the general applicability of information theoretic methods to the study of all systems composed of many individual parts has been long understood,\textsuperscript{[42, 43]} generating insights into the functioning of economic systems from such an application requires a deliberate mapping from competitive actions and economic processes to informational measures we may hope to observe: an informational accounting.

Competitive capitalist enterprises formulate plans to maximize the return on their assets. Their efforts to boost profitability are multifaceted, and involve a wide range of actions that are unobservable or at best difficult to observe: attempts to discover or create new products, production techniques, markets, or new ways to shape market or regulatory conditions; as well as actions involving the movement of capital to existing industries enjoying higher measures of profitability, or trying to follow the productive, managerial, and marketing practices of more profitable firms. We term the former attempts entrepreneurial or innovative, and the latter emulative or arbitrage
actions. By shifting resources and competitive efforts toward undertakings enjoying comparatively high measures of profitability, arbitrage tends to lower rates of return on those undertakings while potentially increasing those of previously less profitable ones now facing less competition. This tends to concentrate or organize measures of profitability amid broader, complex patterns of competitive interaction.

Neither enterprises nor economic observers can know what the effect of any given competitive action on measures of profitability will be. That depends on a complex range of interactions between efforts of many enterprises, as well as exogenous changes to broad economic conditions arising independently from their actions. But the measure of profitability \( r_i \) realized by an enterprise relative to other realized measures of profitability offers an informative quantification of the extent to which the enterprise has been successful in its plans. That measure guides further competitive actions. As a result, the distribution of profitability does not simply reflect competitive actions. It also shapes them. It is an observable social structure that plays a functional role in competitive processes. Its statistical characteristics convey important information about the emergent, macroscopic outcomes of competition in decentralized capitalist economies.

The strong peakedness in that distribution supports the Classical contention that the mobility of capital and of broader competitive efforts toward higher yielding undertakings give rise to a general rate of return on capital, \( r_g \), against which all other realized rates are measured. As such, this rate emerges as the measure of the opportunity cost of capital—the rate investors regard as typical and demand on their investments. Deviations from this rate, \( x_k = r_k - r_g \), offer a measure of excess returns. Its absolute value, \( |x_k| \), offers a measure of the aggregate arbitrage profit realized when a marginal unit of capital is profitably reallocated between a specific undertaking earning \( r_k \) and a typical one earning \( r_g \).

To consider formally how the pursuit of arbitrage profits and other competitive interactions shape the distribution of \( x_i \), let \( \mathcal{E} \) be an exhaustive description of the state of an economic system. This state may be partially represented by distributions of profitability across its competitive enterprises \( f \), defined over a coarse gaining of the support for \( r_i \) into \( k = 1, \ldots, s \) bins. Let \( \Phi = \{ f | \sum_{k=1}^{s} f_k = 1, f_k \geq 0 \} \) be the convex, \( s - 1 \) dimensional probability simplex containing all such distributions. It is possible and useful to consider the dynamic evolution of a competitive economic system \( \mathcal{E}_t \) through the prism of the evolution of the entropy of \( f \), \( S_t \),

\[
\dot{S}_t = \dot{S}_i [t] - \dot{O} [t]
\]  

(4)

where \( \dot{S}_i [t] \) represents the rate of entropy production due to irreversible processes, and \( \dot{O} [t] \) represents the rate of gross entropy reduction taking place in the system.

While we cannot observe individual competitive actions and interactions, and competitive agents themselves cannot know what the impact of their interventions on their profitability and that
of competitors will be, basic combinatorial considerations help cast light onto the systemic or macroscopic effects of different types of competitive actions and shifts in economic conditions. We know from Information Theory and Statistical Mechanics that the likelihood of “accidental” reductions in the entropy of $\mathbf{f}$ is overwhelmingly small for large-$N$ systems. As is well known, a reduction in entropy $-\Delta S$ results in a distribution whose statistical weight is $e^{-N\Delta S}$ times that of the original distribution.\[44\] It is extremely unlikely that a set of generic or not otherwise specified economic developments, like entrepreneurial innovations or exogenous economic shifts, results in entropy reductions in the competitive system.

The only class of economic processes that can be statistically expected to reduce entropy are those driven by their central economic logic to do so: the large collection of individual arbitrage interventions by agents seeking the pecuniary returns that may be realized as long as there is heterogeneity in the distribution of profitability. Gross reductions in the entropy of that distribution are an emergent, unintended outcome of large numbers of such interventions. In the resulting interaction between organizing and disorganizing competitive and broader economic processes, it is the actions of arbitrageurs that collectively shape the patterns of organization we observe in the central regions of distributions of profitability. To understand that shape, it is necessary to consider the economic calculus conditioning those actions, their outcomes and informational moments.

A final point on the appropriate measure of entropy is necessary before proceeding to such consideration. As mentioned above, the observed stretched-exponential distributional forms maximize $(1,d)$ entropy functionals like those in equation 3 above. In line with recent contributions,\[24\] we take this to signify that competitive interactions in markets for goods, labor, and capital, create persistent interdependences between individual values of $x_k$ that effectively reduce the measure of the overall phase spaces available to competitive economic systems in a way that ensures those functionals are the correct measure of the multiplicity of possible micro-configurations supporting a given macroscopic state $\mathbf{f}$. Formally, those interactions ensure that the dependence of those phase-space volumes on the number of agents in the system, $\Omega = \Omega(N)$, takes the limiting form, $\lim_{N \to \infty} \frac{\Omega(N)}{\Omega(N)^\frac{1}{d}} = d$.

Spontaneous reductions in $(1,d)$ entropy are as overwhelmingly unlikely as reductions in Shannon entropy, ensuring that organization in the distributions is still effectively driven exclusively by the actions of arbitrageurs seeking profits available from heterogeneity in those distributions.

4.2 The Formation of $r_g$, Social Scaling, and Entrepreneurial Quasi-Rents

In Classical Political Economy, $r_g$ was broadly understood as the “average” rate of profit.\[12\] In more recent contributions,\[45, 14\] it has been taken as the outcome of more general averaging processes, $r_g = \langle ar \rangle$, defined by weighting vectors $\mathbf{a}$ with $\langle a \rangle = 1$. This allows for the economically meaningful possibility that what economic actors deem general or “typical,” while related to measures of centrality in the distribution of profitability, may not be equal to its first moment.
The dependence of $r_g$ on individual measures of profitability in the economy creates further complex interdependencies between enterprises. In economic terms, it ensures the cost of capital an enterprise faces is competitively shaped by the rates of return realized by all other enterprises that may also be funded in capital markets. Individual values of $x_k = r_k - \langle ar \rangle$ are interrelated, posing additional difficulties for analyses based on detailed descriptions of individual evolutions of values for that quantity. At the same time, those interdependences can also ensure that distributions of $x_k$ are subject to a fairly simple systemic regulation that is implicit in the distributions we observe. Such measures of excess returns embody a *social scaling* of individual measures of profitability.[14]

Many forms taken by this scaling may impose a first-moment constraint on the distribution of $x_k$, so that, over some time periods,

$$\langle x \rangle_t = \langle r \rangle_t - \langle a, r \rangle_t = b$$

This will be true over any time period during which $\langle a, r \rangle$ is stable. In this case, $b > 0$ would reflect a statistically stable measure of underweighting of higher measures of profitability by arbitrageurs in the formation of what they deem to be “typical” returns. Economically speaking, there are at least three reasons why capital-market competition may exhibit such underweighting. The first one is that entrepreneurial innovations may succeed in yielding high measures of profitability more often than they fail. If successful innovations are often difficult to emulate, at least over some time horizons, capital markets may effectively regard the rates of return they generate as less representative of the average opportunities facing investors. A second possibility involves the presence of risk aversion. Undertakings with greater expected variability in their returns may attract less competition, resulting in comparatively higher average returns. As a result, we may expect more enterprises with riskier returns at higher measures of $r_k$. Finally, the high returns of monopolistic firms may be deemed as less representative as those of other enterprises.

![Time Series of Parameter b: Mean](image)

Figure 5: Estimated values for $b$, Germany, France, UK, Italy, and Spain, 2006-2015.
Any kind of underweighting can take a sufficiently statistically stable form to ensure 5 holds for positive \( b \), ensuring the distribution of \( x_k \) has a right skew that partly reflects the presence of \textit{entrepreneurial quasi-rents}—temporary atypically high returns associated with innovation or risk, appropriated by incumbents during the dynamic competitive regulation of profitability around \( r_g \).[46] The skew may also partly reflect the presence of monopolistic rents—high returns in undertakings not immediately open to most investors and, hence, not deemed very representative of typical returns.

4.3 Aggregate Returns and Costs of Arbitrage

We cannot observe individual arbitrage interventions or estimate the returns and costs they pose to those undertaking them. But we can reasonably suppose that in competitive capital markets populated by alert, profit-seeking agents, self-interest and price signals will tend to enforce observable outcomes reflecting the maximization of aggregate arbitrage profits.[25, 35, 47] Those are given by gross arbitrage returns net of the costs arbitrageurs incur in order to realize them. It is this emergent, competitive reckoning of aggregate arbitrage returns and costs that shapes the stretched-exponential distributions we observe.

To characterize this formally, consider a brief lapse of time \( [0, \tau] \) during which competitive interactions take an economic system from a macroscopic state \( E_0 \) containing an original profitability distribution \( o \) with entropy \( S_0 \), to a final state \( E_\tau \) containing a distribution \( f \) whose entropy is \( S_f \). Integration of equation 4 assures us that the entropy change during any such time lapse is given by \( S_o - S_f = \Delta S_e - \Delta O \). Put differently, the total gross entropy reduction effected by the actions of arbitrageurs over this period is given by,

\[
\Delta O = (S_o + \Delta S_e) - S_f
\] (6)

Note that the quantity inside the parentheses in 6 is independent from the actions of arbitrageurs and from the values taken by the system’s distribution during the time period in question. \( S_0 \) is given at the start of the period; and total entropy production is a consequence of entrepreneurial discoveries and interventions—which proceed at their own idiosyncratic pace—and, perhaps most significantly, of exogenous changes in the conditions of the competitive system.

Whatever the details of the dynamic evolution of the competitive system over the brief time period in question, the total, aggregate measure of gross returns realized by arbitrageurs is overwhelmingly likely to vary negatively with \( \langle |d| \rangle_f \)—the average \textit{foregone} arbitrage returns latent in the end-of-period distribution \( f \). Competition can be understood to condition outcomes that minimize this moment of the distribution of profitability, given all other systemic considerations shaping the distribution of \( x_k \).

A characterization of the aggregate costs incurred by arbitrageurs over the period in question
requires a more deliberate exposition. Those costs are defined economically by expenditures necessary to identify and pursue those actions, as well as by the risks they pose. Despite our inability to observe them, we can characterize their aggregate measure informationally, based on a simple conclusion: It is overwhelmingly likely that as arbitrageurs effectively seek to minimize $|x|$, the total costs of the large number of actions they collectively take are increasing on the gross entropy reductions those interventions effect on the distribution of profitability.

In considering this relationship it is important to note that the economic processes involved in the regulation of $x_k$ in each tail of the overall distribution of RoC are well known to be different. Secondary markets for capital goods are notoriously problematic, typically creating stiff transactions costs for parties wishing to liquidate positions in fixed assets.[48] At the same time, competitive pressures are likely to be stronger in left tails since ongoing losses relative to $r_g$ can result in the death of enterprises. The movement of capital into high-yielding undertakings may also pose distinctive costs, including as a result of strategic, deterrent actions by incumbents.[49, 50] This ensures that, in general, the costs and effects of arbitrage interventions, and the character of competitive interdependences between individual measures of $x_k$ are different across each tail. As a result, it is necessary to consider the relationship between total costs of arbitrage interventions and the informational gains they bring about separately for each tail in the distribution.

Within each tail, total costs $C$ will be given by the time integral of the instantaneous total arbitrage costs incurred at each $t \in [0, \tau]$. The latter, in turn, are given by the sum of the individual costs $c_i$ of all arbitrage interventions undertaken under state $E_t$, which we represent by a set of actions $A[E_t]$ containing a large number $i = 1, ..., a[E_t]$ of individual interventions. Formally,

$$C = \int_0^\tau \sum_{i=1}^{a[E_t]} c_i[E_t] \, dt = \int_0^\tau \hat{c}[E_t] a[E_t] \, dt$$  \hspace{1cm} (7)$$

Where $\hat{c}[E_t]$ denotes the average cost per arbitrage intervention undertaken at time $t$.

Along similar lines, the total gross entropy reduction taking place during this period, $\Delta O$, is given by the integral of the instantaneous gross entropy reductions jointly effected by the actions in $A[E_t]$. The latter reductions are the unintended consequence of the actions of arbitrageurs, and hinge on the state of the economy and on the set of arbitrage interventions in the tail in question during time $t$. Formally, denote this by $\hat{O}[t] = \hat{O}[E_t, A_t]$, so that,

$$\Delta O = \int_0^\tau \hat{O}[E_t, A_t] \, dt = \int_0^\tau \hat{o}[E_t, A_t] a[E_t] \, dt$$  \hspace{1cm} (8)$$

Where $\hat{o}[E_t, A_t]$ is the average effected gross entropy reduction per arbitrage intervention undertaken at time $t$.

It is clearly impossible to characterize $C$ and $\Delta O$ fully. That would require detailed knowledge not only of the functions $A$, $a$, $c_i$, and $\hat{O}$, but of the entire path traced by $E_t$ during the period
in question. But so long as the mean average cost and mean average entropy reduction across all possible paths traced by $E$ are well-defined and positive, the Law of Large Numbers ensures that it is overwhelmingly likely that higher measures of $\Delta O$ occur side-by-side with higher total pecuniary costs $C$—since both require a larger number of arbitrage interventions over the time period.

As a result, total costs of arbitrage interventions within a tail can be described by a cost function $C[\Delta O]$ with $C'[\Delta O] > 0$. We also suppose the function is either weakly convex or not "too" concave on $\Delta O$ that it becomes concave on $f$, despite the convexity of $-S_f$.

5 The Systemic Model

We now characterize formally the end-of-period distributions $f$ as the systemic, macroscopic result of the competitive actions of arbitrageurs over the time period in question. We suppose that competition creates tendencies for the exercise of arbitrage to result in a reckoning of gross returns arbitrageurs realize over the period in question with the gross aggregate costs they incur. Arbitrageurs raise funds or have obligations to the same capital market. Competitive capital-market prices (or rates of return demanded by liability holders) faced by would-be arbitrageurs will tend to reflect the demand of all would-be arbitrageurs for funds to support their actions. Competition will thus tend to result in the undertaking of arbitrage operations that maximize net aggregate arbitrage returns or, conversely, minimize aggregate arbitrage losses.

In each tail of the distribution, aggregate arbitrage costs are taken as rising in the measure of informational gains achieved during the time period. Differences in the economic processes responsible for the regulation in each tail ensure that the aggregate arbitrage cost functions in each tail are in general different, and that the parameter $d$ in the $(1,d)$ entropy functional describing the multiplicity of each tail distribution is generally different as well. Formally, this means those costs can be expressed in relation to the end-of-period distribution $f_n$ over infra-modal bins $x_i$, and to distributions $f_s$ over supra-modal bins $x_j$, according to the increasing, convex functions,

$$C_n = C_n \left[ (S_n^0 + \Delta S_n^i) - S_{d_n}(f_n) \right] ; \quad C_s = C_s \left[ (S_s^0 + \Delta S_s^j) - S_{d_s}(f_s) \right]$$ (9)

Finally, we suppose that as the actions of arbitrageurs tend to bring measures of $x_i$ and $x_j$ toward zero, they do so without significantly changing the overall distribution of enterprises between left and right tails of the distribution. The share $a \in [0,1]$ of enterprises with $x_k \geq 0$, and the share $(1-a)$ of those with $x_k < 0$ is understood as determined by other interventions and economic developments, and taken as given to the economic processes regulating the aggregate results of arbitrage interventions.

The distributions we observe may be thus understood as the result of the following piecewise optimization problem, defined over the partial frequency $f^p$ describing the relative occupancy of $x_i$.
bins, and the partial frequency $f_i$ describing the occupancy all bin values $x_j$, 

$$\min_{f_n, f_s} \langle |x| \rangle_f + C_n [A_n - S_{d_n}(f_n)] + C_s [A_s - S_{d_s}(f_s)]$$

subject to $\langle x \rangle_f = b,$

$$||f|| = (1 - a), \quad ||f|| = a$$

(10)

Where $a$, $A_n$, and $A_s$ are given to the process. The associated Lagrangian for this problem is,

$$L = \langle |x| \rangle_f + C_n [A_n - S_{d_n}(f_n)] + C_s [A_s - S_{d_s}(f_s)] - \gamma (\langle x \rangle_f - b)$$

$$+ \mu_n (||f_n|| - (1 - a)) + \mu_s (||f_s|| - a)$$

(11)

with all $\mu_i > 0$, and $\gamma \in [-1, 1]$.

Optimization of this Lagrangian yields a piecewise optimal distribution,

$$f_i = \exp \left[ 1 + \left( \frac{(1 + \gamma) x_i - \mu_n}{\delta_n} \right)^{\frac{1}{\gamma}} \right] ; \quad f_j = \exp \left[ 1 - \left( \frac{(1 - \gamma) x_j + \mu_s}{\delta_s} \right)^{\frac{1}{\gamma}} \right]$$

(12)

Where $\delta_n = C_n'(A_n - S_{d_n}(f_n^*))$ and $\delta_s = C_s'(A_s - S_{d_s}(f_s^*))$ are the marginal costs of additional informational gains in each tail at the optimal $f^*$. These two functions are also subject to normalization constraints,

$$\frac{1 + \gamma}{\delta_n} = \frac{d_n e}{1 - a} \Gamma \left( d_n, \frac{\mu_n}{\delta_n} \right) ; \quad \frac{1 - \gamma}{\delta_s} = \frac{d_s e}{a} \Gamma \left( d_s, \frac{\mu_s}{\delta_s} \right)$$

(13)

It is trivial to show that these two expressions are equivalent to 1 and 2, with,

$$\frac{1}{\lambda_n} = \frac{\delta_n}{1 + \gamma} \equiv \kappa_n ; \quad \frac{1}{\lambda_s} = \frac{\delta_s}{1 - \gamma} \equiv \kappa_s$$

(14)

where $\kappa_i$ represent the effective marginal aggregate cost of informational gains within tail $i$ in the distribution. It is conditioned by the expenditures arbitrageurs needed to undertake to increase the measure of organization in that section of the distribution, and by the losses arbitrageurs face as a result of rents and quasi-rents ensuring some high-return projects are difficult to emulate.

The model in equations 12 and 13 accounts for double stretched-exponential distributions of RoC neither as the result of would-be “representative,” independent drift-diffusion evolutions, nor as expressions of the cognitive or broader characteristics of any individual. It does not based on a systemic characterization of the competitive outcome of the actions of arbitrageurs. The distributions are thus taken to reflect the economic calculus capital-market competition imposes on the set of all agents capable to undertaking emulative or arbitrage interventions. They are the outcome of the aggregate minimization of losses associated with the profit-seeking actions of
arbitrageurs, and the losses imposed on them by the presence of rents and dynamic quasi-rents among enterprises earning high returns.

The model defines a series of macroscopic measures that characterize different systemic features of competitive systems. The parameters $d_n$ and $d_s$ offer informational quantifications of the effect of general competitive interdependences on the set of all profitability measures; $b$ offers a systemic measure of the significance of rents and quasi-rents in the economy; and $\kappa_n$ and $\kappa_s$ represent the effective pricing of informational gains arising from the competitive actions of arbitrageurs in the presence of those rents and quasi-rents. These measures may help inform the understanding of the systemic effects of competition and the formulation of competition policy.

6 Implications and Discussion

The model’s success also points to at least three promising lines of further theoretical work.

This first line of work relates to the broader applicability of the informational characterization of aggregate competitive outcomes offered by the paper. The approach to the systemic regulation of profitability may have general applicability to other quantities subject to regulation by arbitrage. An interesting instance that supports the approach taken above involves distributions of “Tobin’s $q$,” a measure of the valuation of a corporation’s liabilities or securities relative to measures of the value of its assets. Those distributions are strongly organized around modal values. The simplest expression of that organization is the Asymmetric Laplace form it takes for the logarithm of $q$.[51] Notably, the difference between the logarithm of a corporation’s $q$ and the logarithm of the $q$ value for the “typical” or modal corporation is a measure of pecuniary returns on arbitrage; in this case the capital gains a corporation’s managers can realize by arbitraging between the effective cost of capital (given by the total rate of return on assets investors expect of the “typical” corporation) and the total rate of return on assets investors expect on its assets.[45] Here too, the Asymmetric Laplace distribution may be taken as conditioned by the aggregate tradeoff between gross arbitrage profits and the costs imposed by organization in the distribution of $\log q$. Its shape expresses the effective pricing of information arising from the competitive actions of arbitrageurs. This type of pricing may be a general feature of quantities regulated by competitive arbitrage, and of the empirical manifestation of the Law of One Price.

The characterization of distributions of measures of profitability as a statistical equilibria opens the way for a new and distinctively Classical understanding of the content and competitive determination of prices.

All traditions in economic analysis agree that prices are most immediately conditioned by the interaction between supply and demand flows for the goods in question. Most of them predicate “natural” prices on sets of deterministic market equilibria. Where they generally differ is in their understanding of what ultimately conditions supply and demand for goods, which reflects different
appreciations of the most general determinants and the content of equilibrium prices. Contempo-
rary microeconomics predicates competitive prices on Walrasian general equilibria.\[31\] The supply
and demand behavior defining those equilibria are predicated on strongly specified descriptions
of the technological constraints facing individual firms and of the subjective preferences of indi-
vidual consumers over bundles of goods. But production techniques and consumer preferences
are continuously shaped and redefined by the competitive efforts of enterprises, posing significant
conceptual problems for this parametrization of the competitive process: It predicates prices estab-
lished by competition on individual characteristics that are themselves evolving as part competitive
processes.\[25\]

Classical Political Economy takes a different approach. Because profits are most immediately de-
defined by an enterprise’s ability to sell their outputs at prices that exceed their expenditures on labor
and on inputs purchased from other enterprises, it is possible to think of prices as part of economy-
wide structures of “prices of production”: Prices predicated on wage structures, input-output
productive relationships, and on the measures of profitability they define for each enterprise.\[18,
19\] This approach is more general than Walrasian ones because it considers that the mobility of
capital and broader competitive efforts regulate the evolution of supply and demand flows, giving
rise to prices that equate measures of profitability across all undertakings in the economy. Un-
fortunately, this contains the implausible assumption of persistent, deterministic capital-market
equilibria, creating serious conceptual and empirical difficulties.\[2, 5\] and \[52\]

The findings reported here point to the usefulness of a generalized Classical approach that looks
to the mobility of capital and broader competitive efforts as the most general determinant of the
evolution of the structure of generalized prices of production in a capitalist economy: The dynamic
evolution of those structures is shaped by a broad range of changes affecting an economy, including
entrepreneurial efforts to innovate production techniques and to change preferences, market shares,
and other conditions in input and output markets. But the observed outcomes of those complex
competitive interactions also reflect the movement of capital and broader competitive efforts to
higher yielding undertakings. This competitive movement exerts a general and persistent influence
on supply and demand conditions across most markets, shaping the structure of prices. Those
structures necessarily reflect the persistent macroscopic regularities in the distributions of prof-
itability. The relationship between competitive price structures, wages, and productive techniques
is mediated by the capital-market statistical equilibria. this Letter identified. The ensuing relations-
ships hold independently of whether or not markets for goods are at equilibrium. This conclusion
can usefully inform further observational work into the functioning and distributional content of
decentralized capitalist economies.

Finally, the paper’s discussion may have even wider application to analysis of social and other
systems capable of self-organization over some of their functional domains. When considering the
functioning of many such systems we know that entropy reductions are costly, in the sense that
they are not spontaneous and require some form of input to the system. But we often do not have anything like the clear relationship that exists between entropy reduction and its minimum energy input costs in thermodynamic systems.[53] Despite this, the paper has shown that if gross entropy reductions in distributions representing a system’s state over some of its functional domains are the result of large numbers of individual processes, there are fairly general conditions that ensure a weaker result holds: That over any given time period, greater measures of gross entropy reduction have greater input requirements. This applies to any such system, no matter the nature of its functioning, of the processes reducing its entropy, or the kind of input needed to achieve entropy reductions. It is simply a statement that across thermodynamic, social, or cognitive systems, we should expect the rejection of greater volumes of their available phase spaces to require greater aggregate measures of the relevant input.

This parsimonious conclusion can be very informative in analysis of social, economic, and cognitive systems, whose functioning can often be characterized in relation to the effective pursuit of objectives or goals. In many instances, those goals may be characterized as the minimization of a moment of frequency distributions describing part of the macroscopic state of the system in question. Such minimizations require entropy reductions, creating a tension and tradeoffs between pursuit of the system’s objective and its informational costs. Informational costs may have a variety of different expressions. They may appear as a constraint on measures of informational association between distributions representing a system’s inputs and outputs.[38] They may appear as a lower bound on the entropy of outcomes of individual cognitive processes.[54, 9] Or they may appear as an explicit, positive schedule of costs posed by different measures of organization in a system’s state—*inter alia*.

In all cases, the resulting tradeoffs will shape potentially observable distributions describing the macroscopic state of those systems over the domains in question. As in the present case, the observed distributions express a rather simple result—an effective pricing by the system in question of informational gains, measured in terms of the quantities defining its effective goals. This pricing reflects the influence of basic combinatorial or informational realities in the determination of all manner of systemic outcomes.

**References**


