

Deterministic noise amplifiers

Robert J. Deissler^{a,1} and J. Dooyne Farmer^{b,2}

^a*Center for Nonlinear Studies, MS B258, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

^b*Complex Systems Group, Theoretical Division, MS B213, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

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Although we are accustomed to considering only overall stability properties of dynamical systems, local stability properties can also have a dramatic physical effect. Local instabilities in overall stable motion can cause otherwise imperceptible ambient noise to be amplified to macroscopic proportions. The result can be easily mistaken for deterministic chaos. We give measures for local instability and noise amplification in terms of the covariance matrix and local divergence rates and analyze several examples. An experimental test can be made by varying the external noise level; for sufficiently small amplitudes, the noisy response due to local instabilities scales linearly with the noise level, whereas noisy behavior due to deterministic chaos is largely unaffected.

1. Introduction

What is the underlying cause of fluid turbulence and other irregular, unpredictable, aperiodic, or “noisy” phenomena? We are currently offered two choices: One of these, deterministic chaos^{#1}, generates “noise” directly from intrinsic dynamics, without any need for external perturbations. The other choice, “stochastic behavior”, is more vague; typically this phrase is applied to phenomena such as thermal motion, where the dynamics are so complicated as to preclude any detailed description [2]. In experimental studies of noisy phenomena such as fluid turbulence, the question often arises: Is aperiodic behavior intrinsic to the macroscopic equations of motion, or

is it caused by external stochastic influences, such as thermal motions or fluctuations in the boundary conditions^{#2}?

This paper expands on ideas presented in a previous paper [3], which in turn expanded on ideas presented in a paper by Shaw (appendix in ref. [1]). The central point is that the answer to the question posed above is not necessarily clear-cut. A dynamical system can strongly amplify stochastic influences initiated from the external environment without spontaneously generating “noise” or “chaos” of its own accord. The result is noisy behavior that would not be observable without the action of deterministic dynamics, but cannot be sustained by deterministic dynamics alone. One mechanism by which this can happen

¹Present Address: ICOMP, MS 5-3, NASA Lewis Research Center, 21000 Brookpark Rd., Cleveland, OH 44135, USA.

²Present Address: Prediction Company, 234 Griffin St., Santa Fe, NM 87501, USA.

^{#1}For a review on the subject of deterministic chaos, see, e.g., ref. [1].

^{#2}*External* in this sense means, “external to the system under consideration”; if we want to consider properties of the Navier–Stokes equations, for example, thermal motions are “external” to our system, even though the fluid is made of molecules.

is straightforward: Local instabilities in phase space cause fluctuations to be temporarily amplified, even though deterministic motion is quite stable when averaged over the long term. Although it is obvious that this can occur, this phenomenon has received very little study. This is dangerous, since such behavior can be easily confused with deterministic chaos. There is, however, a simple test to distinguish the two types of behavior which should be implemented in experiments where the ultimate source of aperiodic behavior is in question.

A point in a region of local instability can be strongly influenced by external noise. Perhaps the most familiar case where this occurs is deterministic chaos. As pointed out by Shaw [1], positive Lyapunov exponents imply noise amplification, and the spontaneous production of new macroscopic information through the amplification of small fluctuations. Existence of deterministic chaos is not, however, necessary for this to occur. If an instability is of sufficient strength and duration, departures from a deterministic trajectory may be dramatic, causing unpredictable, chaotic-looking behavior, even on a deterministic trajectory that is asymptotically stable. Note, though, that this requires *ongoing* fluctuations, since when motion is asymptotically stable small perturbations of initial conditions are unimportant. Thus, such behavior might be called “sensitive dependence to noise without sensitive dependence to initial conditions” [3]. Deterministic chaos, in contrast, produces sensitive dependence to both noise *and* initial conditions. Deterministic chaos persists in the limit where the external noise goes to zero, spontaneously generating irregular behavior without outside help, whereas instabilities that are merely local have no effect on purely deterministic motion. Nevertheless, it is possible for local instabilities to amplify otherwise imperceptible fluctuations, creating perceptible macroscopic fluctuations that can be easily mistaken to be the result of deterministic chaos.

Before giving some examples we now define more precisely what we mean by local instability.

2. The covariance matrix, local divergence rate, and noise amplification

Consider a dynamical system F . Assume the random influences on F are additive, so that the system under consideration is of the general form

$$\dot{x} = F(x, t) + \sigma \cdot \eta(t), \quad (1)$$

where $\sigma \cdot \eta(t)$ represents ongoing external random perturbations. We assume that each component of η is a Gaussian white noise with a standard deviation of 1. In component form eq. (1) is written

$$\dot{x}_i = F_i(x_j, t) + \sum_{j=1}^d \sigma_{ij} \eta_j(t), \quad (2)$$

where d is the dimension of the system. The equation for the unperturbed orbit is

$$\dot{x}_i^{(u)} = F_i(x_j^{(u)}, t) \quad (3)$$

where the superscript (u) refers to the unperturbed solution. Subtracting eq. (3) from eq. (2), and assuming that σ_{ij} is small so that we may linearize, gives for the deviation from the unperturbed orbit $\delta x_i \equiv x_i - x_i^{(u)}$

$$\delta \dot{x}_i = \sum_{j=1}^d \left(\frac{\partial F_i}{\partial x_j} \delta x_j + \sigma_{ij} \eta_j(t) \right). \quad (4)$$

Define the covariance matrix by $K_{ij} \equiv \langle (\delta x_i - \langle \delta x_i \rangle)(\delta x_j - \langle \delta x_j \rangle) \rangle$, where the brackets signify the expectation value or the ensemble average. Since eq. (4) is linear, it may be shown from the theory of stochastic differential equations [2] that K_{ij} satisfies the equation

$$\dot{K}_{ij} = \sum_{k=1}^d \left(\frac{\partial F_i}{\partial x_k} K_{kj} + K_{ik} \frac{\partial F_j}{\partial x_k} + \sigma_{ik} \sigma_{kj} \right). \quad (5)$$

The trace of K_{ij} gives the variance

$$\left(K \equiv \sum_i \langle (\delta x_i - \langle \delta x_i \rangle)^2 \rangle \right)$$

from the deterministic orbit

$$K = \sum_{i=1}^d K_{ii}. \quad (6)$$

Noting that the deviation along the orbit is $\delta u_{\parallel} = (\sum_i \delta x_i \dot{x}_i) / (\sum_i \dot{x}_i^2)^{1/2}$ (i.e. the projection of δx_i in the direction of \dot{x}_i), we find for the variance along the orbit $K_{\parallel} \equiv \langle (\delta u_{\parallel} - \langle \delta u_{\parallel} \rangle)^2 \rangle$

$$K_{\parallel} = \frac{\sum_{i,j=1}^d F_i F_j K_{ij}}{\sum_{i=1}^d F_i^2}. \quad (7)$$

Also, subtracting the component of δx_i along the orbit from δx_i , we find for the variance perpendicular to the orbit

$$K_{\perp} = K - K_{\parallel}. \quad (8)$$

The covariance matrix K_{ij} may be calculated by solving eq. (5) along with eq. (3). Eqs. (6), (7), and (8) may then be used to calculate the variance, the variance along the orbit, and the variance perpendicular to the orbit, respectively, from the deterministic orbit due to the presence of the Gaussian white noise, as long as the magnitude of the noise is small enough to justify the linearization assumption of eq. (4). This then is a deterministic quantity that will clearly show in which regions noise will be amplified. K_{\parallel} will give the amount of diffusion along the orbit and K_{\perp} will give the amount that the orbit is smeared out due to the presence of the noise. We note that even when the amplification is sufficiently large so that nonlinearities are important, eqs. (5), (6), (7), and (8) may still be used to see in which regions noise will be amplified, although the correct magnitude of the amplification will not be given.

Another quantity which is useful in determining in which regions noise will be amplified is the

local divergence rate [4–6] defined by

$$\gamma(t) = \frac{d}{dt} \ln \left[\left(\sum_{i=1}^d \delta x_i^2 \right)^{1/2} \right], \quad (9)$$

where the separation δx_i between two nearby orbits is given by integrating (from eq. (3))

$$\delta \dot{x}_i = \sum_{j=1}^d \frac{\partial F_i}{\partial x_j} \delta x_j. \quad (10)$$

In eq. (9) one may take an orthonormal set of vectors $\delta x_i^{(n)}$ which are the principal axes of a d -dimensional ellipsoid, where each point on the ellipsoid evolves according to eq. (10), to define the local divergence rate in various directions. The time average of these quantities is simply the Lyapunov exponents [7]

$$\lambda_n = \langle \gamma_n(t) \rangle_t = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \gamma_n(t) dt.$$

Explicitly taking the time derivative in eq. (9) and using eq. (10) gives for the local divergence rate in the direction of δx_i

$$\gamma(t) = \frac{\sum_{i,j=1}^d \delta x_i (\partial F_i / \partial x_j) \delta x_j}{\sum_{i=1}^d \delta x_i^2}. \quad (11)$$

Taking δx_i in the direction of \dot{x}_i by letting $\delta x_i = N \dot{x}_i$, where N is an arbitrary normalization constant, and inserting this into eq. (11) and using eq. (3) gives for the local divergence rate along the flow

$$\gamma_{\parallel}(t) = \frac{\sum_{i,j=1}^d F_i (\partial F_i / \partial x_j) F_j}{\sum_{i=1}^d F_i^2}. \quad (12)$$

For a periodic orbit the time average of this quantity is zero corresponding to the largest Lyapunov exponent (i.e. $\lambda_1 = \langle \gamma_{\parallel}(t) \rangle_t = 0$). Also, for a periodic orbit, the time average of the local divergence rate perpendicular to the flow γ_{\perp} gives the second largest Lyapunov exponent (i.e. $\lambda_2 = \langle \gamma_{\perp}(t) \rangle_t$). For the two-dimensional flow $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ we may get an explicit

expression for γ_{\perp} by taking $(\delta x, \delta y)$ in the direction perpendicular to the orbit by letting $(\delta x, \delta y) = N(-\dot{y}, \dot{x})$, where N is an arbitrary normalization constant. Inserting this into eq. (11) gives

$$\gamma_{\perp} = \left\{ (\partial f / \partial x) g^2 + (\partial g / \partial y) f^2 - [(\partial f / \partial y) + (\partial g / \partial x)] fg \right\} / (f^2 + g^2). \quad (13)$$

In any region for which $\gamma_{\parallel} > 0$ noise will be amplified along the flow increasing the rate of phase diffusion; and in any region for which $\gamma_{\perp} > 0$ noise will tend to be amplified perpendicular to the flow smearing out the orbit in these regions. Although in most cases $\gamma_{\parallel} > 0$ and $\gamma_{\perp} > 0$ may be good indicators for local instability parallel and perpendicular to the orbit, respectively, there may be cases, as noted below, in which $\gamma_{\perp} > 0$ may not be such a good indicator. For this reason we defined local instability in terms of the covariance matrix.

3. Examples

We now consider a few examples to make the above concepts clearer.

Example 1: A fixed point of modulated stability. Suppose F has a fixed point attractor x_0 that is stable on the average, but has temporary periods of instability. During stable periods, the orbit will remain close to x_0 , but the noise $\eta(t)$ will prevent the orbit from ever completely settling onto x_0 . If x_0 temporarily becomes unstable, however, the orbit will exponentially move away with a magnitude and direction that depend sensitively on the position of the orbit at the onset of the instability. When stability resumes, the orbit will return to the vicinity of the fixed point. Thus, instabilities cause "chaotic" pulses, whose amplitude and direction in phase space vary unpredictably.

For example, consider the simple equation

$$\dot{x} = A(t)x + \sigma\eta(t), \quad (14)$$

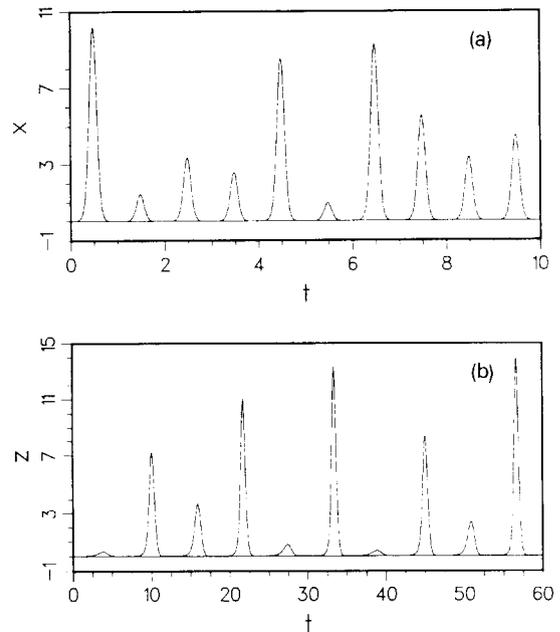


Fig. 1. (a) A typical trajectory of eq. (14) with $A(t) = k \sin(\omega t) + c$, where $k = 26$, $c = 2$, and the noise is introduced by adding a random number uniformly distributed between -0.9×10^{-3} and 1.1×10^{-3} to x at each time step $\Delta t = 0.01$. (b) A typical trajectory of $z(t)$ on the chaotic Rössler attractor, eq. (23), with $a = 0.2$, $b = 0.2$, and $c = 4.6$.

where $A(t)$ is any periodic function with period T such that there are values of t for which $A(t) > 0$, yet $\int_0^T A(t) dt < 0$. For instance, let $A(t) = k \sin(\omega t) + c$, where $c < 0$ and $|k| > |c|$. A typical trajectory $x(t)$ consists of pulses of random height and sign. On the surface this is quite similar to the behavior of some systems with strange attractors, such as $z(t)$ of the Rössler attractor [8]. The resemblance to the Rössler attractor is even more pronounced if the noise is biased so that the fluctuations $\eta(t)$ all have the same sign, as shown in fig. 1.

The resemblance between these two types of motion disappears completely when the external noise is removed. When $\sigma = 0$, $x(t) = 0$ for eq. (14), while motion on the Rössler attractor is virtually unaffected.

From eq. (5) we find that the variance K from the deterministic orbit satisfies

$$\dot{K} = 2A(t)K + \sigma^2. \quad (15)$$

Since we are interested in the asymptotic state, the initial variance is arbitrary. For convenience we take an initial variance of 0, giving as a solution of eq. (15),

$$K(t) = \sigma^2 \exp\left(2 \int_0^t A(s) ds\right) \times \int_0^t \exp\left(-2 \int_0^s A(r) dr\right) ds. \quad (16)$$

It is difficult to see the behavior from eq. (16) without looking at a specific example, so we now consider a special case in which the integrals can be done exactly. Let $A(t)$ be a square wave of amplitude a with an offset $-c$,

$$A(t) = -a - c \quad \text{if } m\tau \leq t < (m + \frac{1}{2})\tau, \\ = a - c \quad \text{if } (m + \frac{1}{2})\tau \leq t < (m + 1)\tau. \quad (17)$$

The purely deterministic system ($\sigma = 0$) has a fixed point at $x = 0$. For $c > 0$, this fixed point is overall stable, and is the global attractor of the system. When $a > c$, the instantaneous stability changes sign at periodic intervals.

For large times we find during the stable phase

$$K_s(t) = B_s(\tau, \sigma) \exp[2(a + c)(M\tau - t)] + \frac{\sigma^2}{2(a + c)}, \quad (18)$$

where $M \equiv \text{Int}(t/\tau)$ ("Int" means "the integer part of") and

$$B_s(\tau, \sigma) = \frac{a\sigma^2}{a^2 - c^2} \left(\frac{\exp(2c\tau) - \exp[(a + c)\tau]}{1 - \exp(2c\tau)} \right)$$

and during the unstable phase

$$K_u(t) = B_u(\tau, \sigma) \exp[2(c - a)(M\tau - t)] - \frac{\sigma^2}{2(a - c)}, \quad (19)$$

where

$$B_u(\tau, \sigma) = \frac{a\sigma^2}{a^2 - c^2} \times \left(\frac{\exp[2(c - a)\tau] - \exp[(3c - a)\tau]}{1 - \exp(2c\tau)} \right)$$

Note that eqs. (18) and (19) satisfy eq. (15) and the matching conditions $K_s[(m + \frac{1}{2})\tau] = K_u[(m + \frac{1}{2})\tau]$ and $K_s[m\tau + \epsilon] = K_u[m\tau - \epsilon]$, where $\epsilon \rightarrow 0$. Although these expressions are messy, the essential features are seen by noting that over the stable phase the variance is damped with damping rate $2(a + c)$ and over the unstable phase the variance is amplified with growth rate $2(a - c)$. Therefore, the result is a sequence of pulses of random amplitude, occurring at periodic intervals. As $\sigma \rightarrow 0$, so does K , and the chaotic-looking behavior disappears.

It is also interesting to look at the case $c = 0$, which represents a diffusion process. In this case we find

$$K_s(t) = \frac{\sigma^2}{a} \exp[2a(M\tau - t)] \times \{[\exp(a\tau) - 1]M - \frac{1}{2}\} + \frac{\sigma^2}{2a} \quad (20)$$

and

$$K_u(t) = \frac{\sigma^2}{a} \exp\{-2a[(M + \frac{1}{2})\tau - t]\} \times [(1 - \exp(a\tau))(M + \frac{1}{2}) + \frac{1}{2}] + \frac{\sigma^2}{2a}. \quad (21)$$

If $A(t) = 0$ we instead have the usual diffusion equation $K(t) = \sigma^2 t$ (see eq. (15) with $A = 0$). The important point to note here is that the diffusion rate can be much larger when there are regions of expansion and contraction. For example, at the phase of maximum amplitude, $t = m\tau$, $K_{\max}(t) = \sigma_{\max}^2 t$, where $\sigma_{\max}^2 = (\sigma^2/a\tau) \times [\exp(a\tau) - 1]$. At the phase of minimum amplitude, $t = (m + \frac{1}{2})\tau$, $K_{\min}(t) = \sigma_{\min}^2 t$, where $\sigma_{\min}^2 = (\sigma^2/a\tau)[1 - \exp(-a\tau)]$. Also, K averaged over

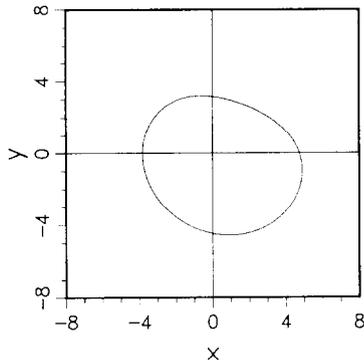


Fig. 2. The Rössler attractor, eq. (22), for $a = 0.2$, $b = 0.2$ and $c = 2.6$. A point on the orbit moves in a counterclockwise direction.

one cycle is $K_{\text{ave}}(t) = \sigma_{\text{ave}}^2 t$, where $\sigma_{\text{ave}}^2 = 2(\sigma/a\tau)^2[\cosh(a\tau) - 1]$. For $a\tau \ll 1$, σ_{max}^2 , σ_{min}^2 , and σ_{ave}^2 all reduce to the usual diffusion rate of σ^2 . However for larger values of $a\tau$, σ_{max}^2 and σ_{ave}^2 can be much larger than σ^2 .

Example 2: A limit cycle with a local instability.

The example above (for $c > 0$) is a limit cycle in disguise. When viewed as an autonomous system of equations, a fixed point of modulated stability becomes a limit cycle with the special property that added noise does not affect the phase. However, in general, local instabilities will amplify noise so as to influence both amplitude and phase.

As an example of a limit cycle we consider the Rössler equations [8]

$$\begin{aligned}\dot{x} &= -(y + z), \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c).\end{aligned}\quad (22)$$

Fig. 2 shows the orbit in the x - y plane. We first solved eq. (5) (where $x \equiv x_1$, $y \equiv x_2$, and $z \equiv x_3$ and with $\sigma_{ij} = \sigma\delta_{ij}$) along with eq. (22) and used eqs. (6), (7), and (8) to calculate the variance K , the variance along the orbit K_{\parallel} , and the variance perpendicular to the orbit K_{\perp} , respectively. Fig. 3 shows the square root of these quantities for twenty orbits starting with an initial variance of zero, where $t = 0$ corresponds to $y = 0$ ($x > 0$). After the system reaches an asymptotic state K

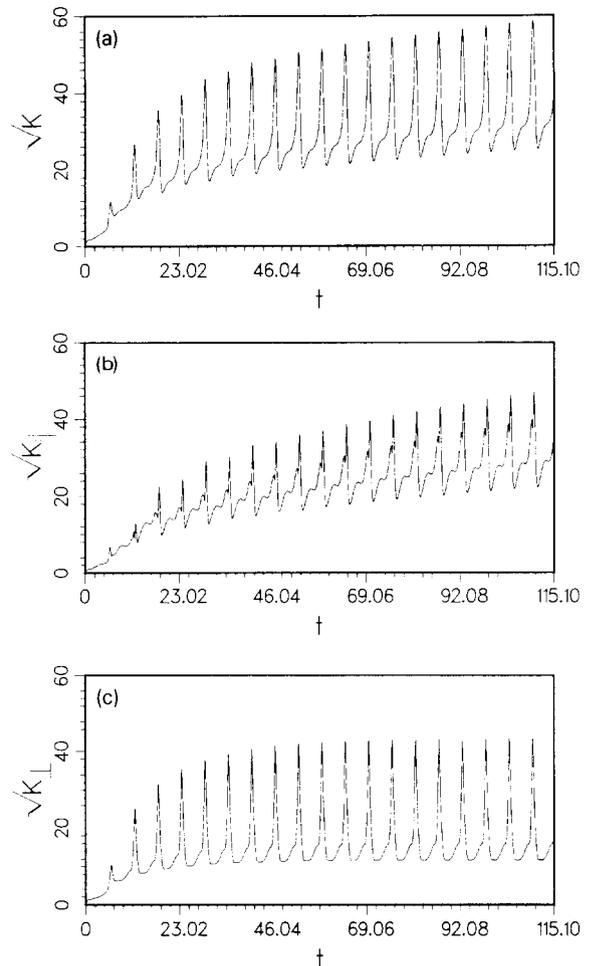


Fig. 3. (a) The standard deviation from a deterministic trajectory in the linearized limit due to the presence of Gaussian white noise with standard deviation $\sigma = 1$ for twenty cycles of the Rössler attractor. $t = 0$ corresponds to $y = 0$ ($x > 0$) in fig. 2. (b) Standard deviation parallel to the orbit. (c) Standard deviation perpendicular to the orbit.

and K_{\parallel} continue to increase linearly on the average corresponding to diffusion along the orbit, whereas K_{\perp} levels off on the average. Note that the diffusion rate is larger in regions where K and K_{\parallel} are large.

We also plot the local divergence rates parallel (from eq. (12)) and perpendicular (by calculating the separation between two nearby states using eq. (10) and subtracting the component along the flow at each time step) to the orbit as a function

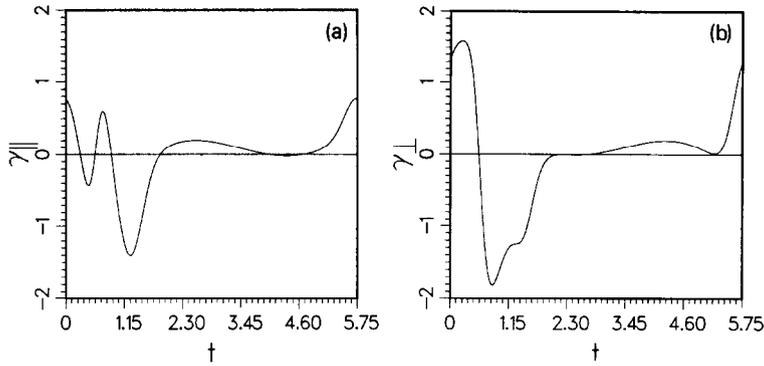


Fig. 4. (a) The local divergence rate parallel to the orbit for one cycle of the Rössler attractor. (b) The local divergence rate perpendicular to the orbit.

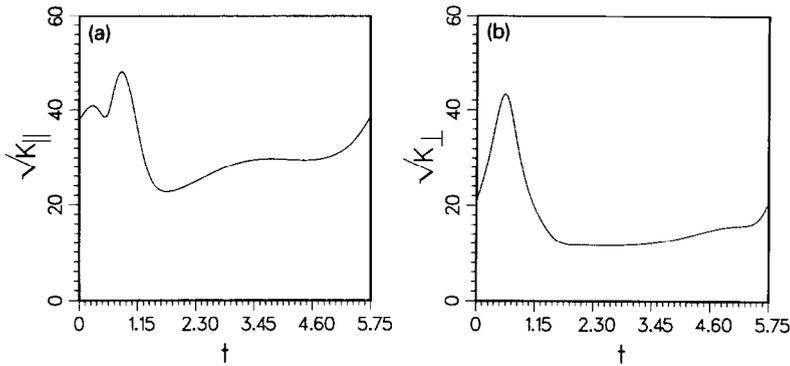


Fig. 5. (a) The standard deviation parallel to the orbit from a deterministic trajectory in the linearized limit due the presence of Gaussian white noise with standard deviation $\sigma = 1$ for one cycle of the Rössler attractor. (b) The standard deviation perpendicular to the orbit.

of time for one orbit. Fig. 4 shows the result. Fig. 5 shows $\sqrt{K_{\parallel}}$ and $\sqrt{K_{\perp}}$ as a function of time for one orbit. Comparing fig. 5 with fig. 4 we see that the variance tends to increase in regions where the local divergence rate is positive. Although γ_{\perp} gives a good measure here for noise amplification perpendicular to the orbit, this will not be necessarily true in general for systems with dimension three or greater. This is so since, if the orbit goes below the noise level (i.e. $K < \sigma^2$), a deterministic trajectory will be entirely lost and even though $\gamma_2 < 0$, γ_n for some value of $n > 2$ may be greater than 0 causing noise to be amplified (see discussion following eq. (10)). In these cases it will be necessary to calculate K_{\perp} which will be a reli-

able measure for noise amplification perpendicular to the orbit.

Referring to fig. 5b we would expect significant noise amplification for the Rössler equations. Figs. 6a and 6b show plots of the z component of the Rössler equations as functions of x and t , respectively, with a Gaussian white noise of standard deviation 10^{-3} added at each time step ($\Delta t = 0.01$). (Note that since the noise is added discretely at each time step, this corresponds to $\sigma = 10^{-3}/(\Delta t)^{1/2} = 0.01$ in eq. (1).) We see that there indeed is significant noise amplification, for without noise the orbit is strictly periodic.

Fig. 7 shows the standard deviation perpendicular to the orbit for one cycle of the Rössler

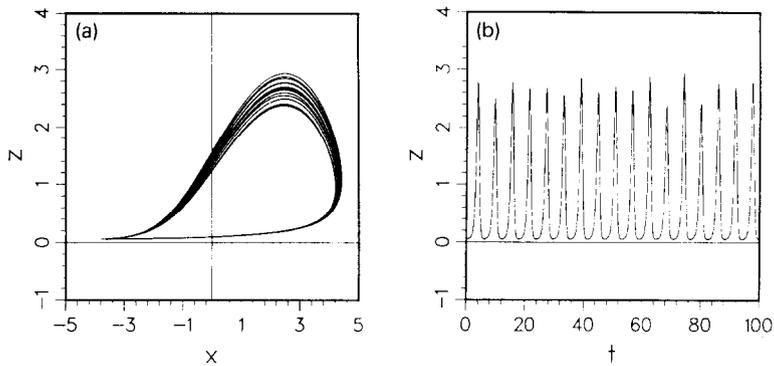


Fig. 6. A typical trajectory of z on the Rössler attractor as a function of x and t , with the same parameter values as those in fig. 2, but with a Gaussian white noise of standard deviation 10^{-3} added to x , y , and z at each time step $\Delta t = 0.01$.

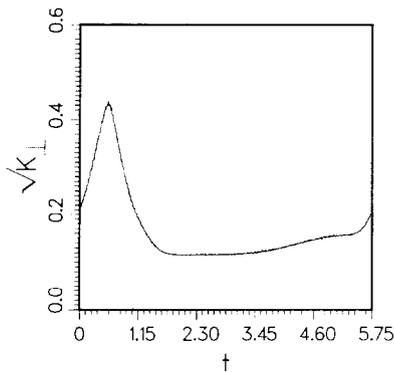


Fig. 7. The standard deviation perpendicular to the orbit from a deterministic trajectory from a direct numerical simulation with the same noise as that in fig. 6 for one cycle of the Rössler attractor.

attractor for the same noise used in fig. 6 from a direct numerical simulation. This was accomplished by creating 1000 bins along the deterministic orbit, binning the noisy orbit in these bins, and calculating the perpendicular distance from the deterministic orbit. Comparing fig. 7 with fig. 5b we see that they agree well showing that the amplitude of the noise is such that the linearization approximation is still nearly valid. We note that the above direct calculation requires a substantial amount of programming and computer time as compared with the much simpler task of solving eq. (5).

Example 3: Statistical limit cycle. Another example is the statistical limit cycle [9, 10]. In modeling the onset of rotating convection Busse derives the following equations.

$$\begin{aligned}\dot{x} &= (1 - x - \beta y - \gamma z)x, \\ \dot{y} &= (1 - y - \beta z - \gamma x)y, \\ \dot{z} &= (1 - z - \beta x - \gamma y)z.\end{aligned}\quad (23)$$

The variables x , y , and z represent three possible orientations of the convection rolls, which make 120 degree angles relative to each other. This same set of equations was originally derived by Leonhard and May [11], but in the context of a population biology problem. They have the remarkable property that for $\beta < 1$ or $\gamma < 1$ and $(\beta + \gamma) > 2$ the attractor is a *heteroclinic cycle*, i.e., the stable manifold of one fixed point is the unstable manifold of another, forming a cycle. A special case, called a *homoclinic cycle* occurs when the unstable manifold of a single fixed point joins smoothly onto the stable manifold. Since this cycle contains a fixed point, in the case of no applied noise the period is infinite. When noise is applied, however, a region of local instability near the fixed point (along the stable manifold) amplifies any external noise, carrying the orbit past the fixed point. The average period of the noisy system is therefore finite. Note that a homoclinic or heteroclinic cycle *necessarily* contains a region of

local instability where the stable manifold is close to the fixed point(s).

4. Comparison to other types of noise amplification

The mechanism for noise amplification discussed above is not the only mechanism for noise amplification. Before proceeding we compare with a few other methods.

Critical phenomena or bifurcations. At a bifurcation parameter value a dynamical system becomes overall neutrally stable. The result is that orbits are “unrestrained”, and in the presence of noise a point in phase space can execute a random walk and thus make large fluctuations [12, 13]. Before proceeding we need to distinguish between a neutrally stable system of constant stability and a neutrally stable system of modulated stability. If the stability is constant in time then the growth rate of the variance is linear with growth rate σ^2 . This behavior is clearly different from the exponential growth of fluctuations over an unstable phase as discussed in this paper. However, if the stability of an overall neutrally stable system is modulated as in eqs. (20) and (21) or as in a periodic orbit at a bifurcation point, the effect studied in this paper is present. Over short times there is exponential growth of fluctuations over an unstable phase, and over long times the growth rate of fluctuations is diffusive, but the diffusion rate can be much larger, as discussed following eq. (21).

Noise induced deterministic chaos. At parameter values where regular behavior occurs, but where there is deterministic chaos nearby, the addition of external noise can cause a transition to chaos. In this case, discussed extensively in ref. [14], the noise effectively acts to shift the bifurcation parameter value so that the transition from periodicity to chaos occurs earlier.

Metastability. Fluctuations whose amplitude exceed a certain critical size can induce transitions between the basins of two attractors, or

between two different wells of a potential [15, 16]. In order to make a transition between two regions of stability it is necessary to pass through a region of instability (assuming the dynamical system is smooth). Thus, in a sense this is a special case of the phenomenon discussed here. An important difference between this and examples 1–4 is that the instability is not on the attractor, so that exponential noise amplification is a rare event and only occurs for perturbations exceeding a certain critical size.

Convective instabilities. In a spatially extended system a localized perturbation can be carried along by the mean flow so that it grows only in a moving frame of reference, eventually damping at any given stationary point. In the deterministic limit there is no motion. If there are ongoing perturbations, however, the resulting behavior downstream in the stationary frame of reference looks very much like deterministic chaos. Three systems that exhibit this behavior have been studied: A one dimensional lattice of quadratic maps with asymmetric coupling [17], the one-dimensional Ginzburg–Landau equation [18], and the Kuramoto–Sivashinsky equation [19]. In all cases, the effect is quite dramatic: In the absence of perturbations the system remains at rest, but the application of an imperceptible amount of noise creates an expanding chaotic-looking “wake” as it propagates downstream.

5. Prevalence, observability, and experimental consequences

In general, there is no reason to expect that local stability will be constant. In the typical case local stability varies from place to place, and there is nothing to prevent the local divergence rate from changing sign. This behavior is likely to be especially common near bifurcations; if the asymptotic local stability is close to neutral, then even small variations in local properties will create instabilities. Prior to the loss of overall sta-

bility, in the general case certain regions will become locally unstable before others.

In order for noise amplification to become observable, however, there are quantitative constraints. In particular, the combination of amplification and external noise must be large enough to generate a detectable signal. The observation of deterministic noise amplification depends on the precision of measurements as well as on the intrinsic dynamics and the level of external fluctuations. Standard examples such as the Rössler equations show noise amplification. Although the effect is not very large it can be quite significant as shown in fig. 6.

The primary motivation of this paper is the interpretation of experimental results. As already demonstrated in fig. 1, local noise amplification can be easily confused with deterministic chaos. In fact, one of the initial motivations for this paper came serendipitously during an experiment with electrical circuits by one of the authors (J.D.F.). In simulating a simple damped harmonic oscillator on an analog computer, rather than relaxing to a simple point in the phase plane, the trajectory sporadically spiraled out. This turned out to be due to a bad operational amplifier. The stability of the amplifier varied with time, producing a circuit that is a second order analog of eq. (14), with periods of time in which the effective damping coefficient changes sign.

In experiments in which it may be suspected that external noise is playing an important role in the observed dynamics, the definitive test is to vary the external noise level and examine the response of the system. Since it is difficult to reduce the noise level in an experiment, the simplest method is to *raise* the noise level by adding additional noise to the system [3].

6. Conclusions

When driven by external noise, local instabilities in otherwise stable dynamical motion can cause behavior that looks quite similar to deter-

ministic chaos. Microscopic fluctuations are thereby amplified to generate irregular *macroscopic* fluctuations in both the amplitude and phase of a signal. In contrast to deterministic chaos, however, chaotic-looking behavior generated by local instabilities disappears when external noise is removed.

It seems quite likely that behavior of the type discussed here is often confused with deterministic chaos. In an experiment where local instabilities are suspected the best method to make the distinction from deterministic chaos is to *add* external noise to the dominant source, and test for linear scaling of the amplitude of noisy behavior with the amplitude of the external noise.

In the wake of the revelations brought about by the discovery of deterministic chaos, the phenomena described in this paper are not very surprising. It is perhaps in part the very simplicity of this phenomena that has caused it to be overlooked. In spite of its simplicity, this behavior is probably common, and the distinction between this and deterministic chaos should be made clear. In any case, the experimental test suggested here should be made in any cases in which there is doubt regarding the source of aperiodic behavior.

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