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An analysis of price impact function in order-driven markets

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Abstract

We introduce a microscopic model of double-auction markets based on random order placement. Traders post market or limit orders which are stored in the book of the exchange and executed via a central order matching mechanism. We use dimensional analysis, simulations and analytical approximations to make testable predictions of the price impact function. We find that the price impact function is always concave, in agreement with empirical measurements. We provide an explanation for its concavity based on the properties of order flows.

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1. Introduction

Electronic trading employing a public limit order book is continuing to gain a greater share of worldwide security trading and many of the major exchanges in the world rely, at least in part, upon limit orders for the provision of liquidity. In order-driven markets investors can submit either market or limit orders. Impatient traders typically submit market orders which are immediately executed against the quoted bid or ask. To guarantee that orders are executed only when the market price is below or above

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a certain threshold, patient traders may prefer to submit limit orders. Limit orders are stored in the book of the exchange and executed using time priority at a given price and price priority across prices. If the market is rising, the upward price movements trigger limit orders to sell; if the market is falling, the downward movements trigger limit orders to buy. Limit orders thus provide liquidity and immediacy to market orders. By delaying transacting, patient traders may be able to trade at a more favorable price. On the other side, limit orders face uncertainty over when and if they will be executed. Also, a trader's perception of what is the fair price of an asset may have changed since the time the order was placed and the order may be canceled. Understanding how the placement of orders contributes to market liquidity is important for practical reasons, such as minimizing transaction costs. Furthermore, it provides a proxy for the demand function. A market is considered liquid if an attempt to buy or sell results in a small change in price. The price impact function, $\Delta p = \phi(\omega, \tau, t)$, where Δp is the logarithmic price shift at time $t + \tau$ caused by a market order of size ω placed at time t, provides a measure of the liquidity for executing market orders. Empirical studies on the NYSE [1] have shown that the immediate response of prices to a single trade is remarkably regular. After appropriate rescaling the data for the 1000 highest capitalized stocks traded in the NYSE collapse onto the same curve. The curve is always concave, with a slope varying from ~ 0.5 to ~ 0.2 depending on stock capitalization.

2. The model

The model described here has been introduced in Ref. [2] and further investigated in Ref. [3]. Related work can be found in Refs. [4–7]. We propose the simple random order placement model shown in Fig. 1. All the order flows are modeled as Poisson processes. We assume that market orders in chunks of σ shares arrive at a rate of μ shares per unit time, with an equal probability for buy and sell orders. Similarly, limit orders in chunks of σ shares arrive at a rate of α shares per unit price and per unit time. We express prices as logarithms and use the logarithmic price $a(t)$ to denote the position of the best ask and $b(t)$ for the position of the best bid. The gap between them, $s(t) = a(t) - b(t)$, is called the spread and $m(t) = (a(t) + b(t))/2$ the midpoint. Offers are placed with uniform probability at integer multiples of a tick size p_0 (also defined on a logarithmic scale; note this is not true for real markets) in the range $b(t) < p < \infty$, and similarly for bids on $-\infty < p < a(t)$. When a market order arrives it causes a transaction; under the assumption of constant order size, a buy market order removes an offer at price $a(t)$, and a sell market order removes a bid at price $b(t)$. Alternatively, limit orders can be removed spontaneously by being canceled or by expiring. We model this by letting them be removed randomly with constant probability δ per unit time. The model considerably simplifies the complexity of the trading process in order to remain analytically tractable. Nonetheless, the coupling between the bid and ask processes (one determines the boundary condition for the other) makes its time evolution non-trivial. Furthermore, the model can be easily enhanced, in future research, to include a non-Poisson order cancellation process and feedback between orders and prices in the attempt to reproduce some of the features observed in real

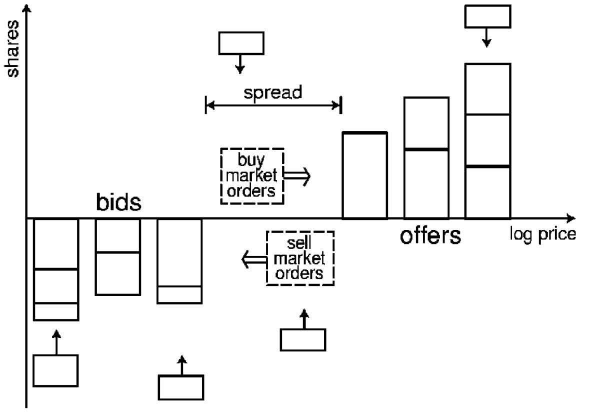

Fig. 1. Schematic of the order-placement process. Stored limit orders are shown stacked along the price axis, with bids (buy limit orders) negative and offers (sell limit orders) positive. New limit orders are visualized as falling randomly onto the price axis. New offers can be placed at any price greater than the best bid, and new bids can be placed at any price less than the best offer. Market orders remove limit orders of the opposite sign, based on best price and earliest time.

data, such as power-law placement frequency [7], and a power-law or log-normal order size distribution.

3. Results

The depth profile $n(p,t)$ gives the number of shares in the order book at price p and time t. We investigate how the depth profile depends on the parameters of the model (see Refs. [2,3] for further results predicted by the model regarding the behavior of prices and spreads).

The simple technique of dimensional analysis allows us to derive several powerful results and simplify the simulation analysis by reducing the number of free parameters from five to two. There are three fundamental dimensions in the model: *shares*, *price*, and *time*. There are also three rate constants: α, with dimensions of $shares/(price \times time)$, μ, with dimensions of $shares/time$, and δ, with dimensions of $1/time$. There are two discreteness parameters: the order size σ and the price tick p_0. The average spread has dimensions of *price* and is proportional to μ/α; this comes from a balance between the total order placement rate inside the spread and the order removal rate. The asymptotic depth is the density of shares far away from the midpoint, where market orders are unimportant. It has dimensions of $shares/price$, and is Poisson distributed with mean α/δ. The slope of the depth profile near the midpoint has dimensions of $shares/price^2$. It is proportional to the ratio of the asymptotic depth to the spread, which implies that it scales as $\alpha^2/\mu\delta$. This is summarized in Table 1. There is a unique way to combine the parameters to create nondimensional units for shares and prices as described in Table 2. When plotting the average depth profiles, obtained by Monte

Table 1
Predictions of scaling of market properties as a function of properties of order flow. α is the limit-order rate, μ is the market-order rate and δ is the spontaneous limit-order removal rate

Quantity	Dimensions	Scaling relation
Asymptotic depth	*shares*/*price*	$d \sim \alpha/\delta$
Spread	*price*	$s \sim \mu/\alpha$
Slope of depth profile	*shares*/*price*^{2}	$\lambda \sim \alpha^2/\mu\delta = d/s$

Table 2
Nondimensional quantities and their definitions from observables with dimensions

Description	Notation	Definition
Depth	$\hat{n}$	$\delta n/\alpha$
Price	$\hat{p}$	$\alpha p/\mu$
granularity	ε	$2\delta\sigma/\mu$

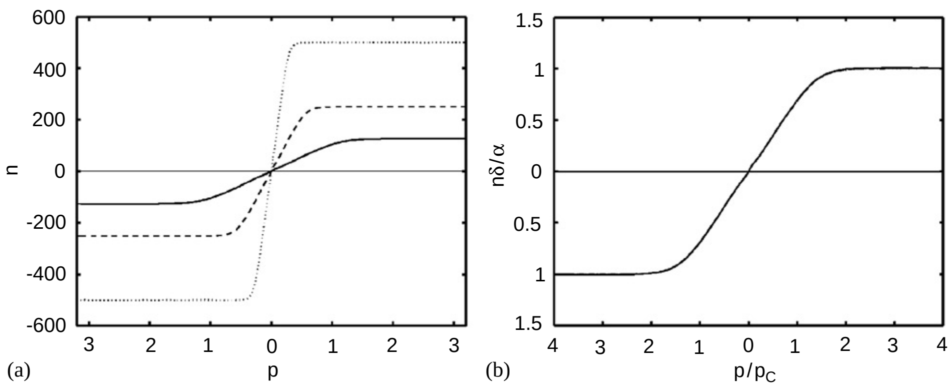

Fig. 2. We plot the average depth profile for three different parameter sets both in (a) dimensional and (b) nondimensional units. The mean order size σ and the tick size p_0 set to one, and $\alpha = 0.5$. p_c is defined as $\mu/2\alpha$. Three simulation results are shown, with $\delta = 0.001$ and $\mu = 0.2$, $\delta = 0.002$ and $\mu = 0.4$, $\delta = 0.004$ and $\mu = 0.8$.

Carlo simulations for different values of δ and μ at fix ε, in terms of nondimensional units, the curves collapse on each other (see Fig. 2). The functional form of the book depth-profile is primarily determined by the granularity parameter ε (Fig. 3a). In the high ε regime the market order removal rate is low and there is a significant accumulation of orders at the ask (bid), so that the average depth $\langle n(p,t)\rangle$ is much greater than zero (note that price increments are calculated from the midprice). In this regime the depth is a concave function of prices. In the intermediate ε regime the market order rate increases, and $\langle n(p,t)\rangle$ decreases, almost linearly, toward zero at the

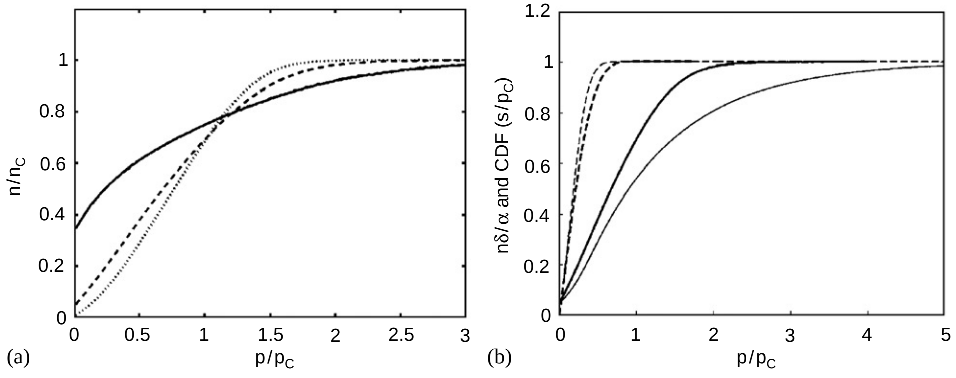

Fig. 3. (a)The mean depth profile, in nondimensional units, for three values of the granularity parameters, $\varepsilon=0.2$ (solid), $\varepsilon=0.02$ (dash) and $\varepsilon=0.002$ (dot). (b) A comparison of simulation results (thin solid line for depth profile and thin dashed line for cumulative distribution of the spread) with the self-consistent solution of the master equation approach (thick solid line for depth profile and thick dashed line for cumulative distribution of the spread) at $\varepsilon = 0.02$.

ask. In the small ε regime the market order rate increases further, and the depth profile becomes a convex function close to the ask.

We have studied the model analytically using a master equation approach and mean field approximations to obtain the solution [2,3]. Analytical results in good qualitative agreement with the simulation both for the depth profile and the cumulative distribution of spreads can be obtained from one or another of these methods over a wide range of parameters (Fig. 3b).

When the fluctuations of the depth profile are small, a good approximation of the instantaneous price impact function is given by inverting the obvious relationship,

$$\omega = \int_0^{\Delta p} \langle n(p,t)\rangle \, \mathrm{d}p \; . \tag{1}$$

As long as the depth profile is monotonically increasing, ω is concave. Expanding $\langle n(p,t)\rangle$ in a Taylor series, one obtains terms $\Delta p = \phi(\omega,t,t) \sim \omega^{\beta}$ from each order, with $\beta \leqslant 1$, which approximate the impact function over the ranges of price dominated at that order. This result is in agreement with the best available empirical evidence. The exponent $\frac{1}{2}$ appears as a special case, when the depth profile satisfies the condition of vanishing at the midpoint and having a derivative that exists so that the depth increases linearly through a sufficiently wide range of prices.

Even if our random flows model contains several unrealistic assumptions, it is because of its simplicity that we can relate its prediction to measurable properties of real markets. We find that the price impact function is always concave, in agreement with the best available empirical measurements, and suggest that concavity emerges from the trading mechanism and market structure rather than rationally optimized trading strategies.

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